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THETA IDENTITIES WITH COMPLEX MULTIPLICATION

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Introduction. This paper grew out from the attempt to refine the notion of a symmetric line bundle on an abelian variety in the case of complex multiplication. Recall that a line bundle L on an abelian variety A is called *symmetric* if $(-\mathrm{id}_A)^*L \simeq L$. It is known that in this case one has an isomophism

$$(n\,\mathrm{id}_A)^*L \simeq L^{n^2}$$

for any $n \in \mathbb{Z}$. Now assume that A admits a complex multiplication by a ring R, that is, we have a ring homomorphism $R \to \mathrm{End}(A) : r \mapsto [r]_A$. If L is nondegenerate, then the corresponding polarization $\phi_L : A \to \hat{A}$ (where $\hat{A}$ is the dual abelian variety to A) defines the Rosati involution on $\mathrm{End}(A) \otimes \mathbb{Q}$ (see [5]). Assume that this involution is compatible with some involution ε on R. Let $R^+ \subset R$ be the subring of ε-invariant elements. Then for every $r \in R^+$, the homomorphism $\phi_L \circ [r]_A : A \to \hat{A}$ is self-dual; hence, one can ask whether it comes from some "natural" line bundle $L(r)$ on A. The word "natural" should mean in particular that the map $r \mapsto L(r)$ from R^+ to the group of symmetric line bundles on A is a homomorphism, resembling the usual homomorphism $n \mapsto L^n$. By analogy with the above isomorphism, we would like to impose the following condition on such a homomorphism

$$[r]_A^* L(r_0) \simeq L(\varepsilon(r) r_0 r)$$

for any $r \in R$, $r_0 \in R^+$. We call such data a $\Sigma_{R,\varepsilon}$-structure (since a suitable generalization of this notion to group schemes with complex multiplication is a refinement of the notion of Σ-structure defined by L. Breen in [2]).

In the first part of the paper we describe an obstruction to the existence of a $\Sigma_{R,\varepsilon}$-structure for a given polarization of A. It turns out that when R is commutative, one can prove the existence of a $\Sigma_{R,\varepsilon}$-structure, assuming that R is unramified at all ε-stable places above 2 (in noncommutative cases, one also needs some additional assumptions at archimedian places). In the case of an elliptic curve E with its standard principal polarization and $R = \mathrm{End}(E)$ this result is sharp: a $\Sigma_{R,\varepsilon}$-structure exists if and only if R is unramified at 2. In the case of commutative real multiplication, one needs only that R is normal above 2 to ensure the existence of a $\Sigma_{R,\varepsilon}$-structure.

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