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THE BOCHNER-RIESZ CONJECTURE IMPLIES THE RESTRICTION CONJECTURE

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1. Introduction Fix $n \geqslant 2$. For any $1 \leqslant p \leqslant \infty$ and $\alpha \geqslant 0$, we use $BR(p,\alpha)$ to denote that $S^{\delta(p)+\alpha}$ is bounded on L^p, where $\delta(p) = \max(n|1/p - 1/2| - 1/2, 0)$ and S^δ is the Bochner-Riesz multiplier

$$\widehat{S^\delta f}(\xi) = (1 - |\xi|^2)_+^\delta \hat{f}(\xi).$$

The Bochner-Riesz conjecture is the statement that $BR(p,\varepsilon)$ holds for all $1 \leqslant p \leqslant \infty$ and $\varepsilon > 0$. Apart from the trivial estimate $BR(2,0)$, this conjecture is optimal (see [12]).

Similarly, we use $R(p,\alpha)$ to denote the localized restriction estimate

$$\|\mathfrak{R}f\|_{L^p(S^{n-1})} \lesssim R^\alpha \|f\|_{L^p(B(0,R))}$$

for f supported in $B(0,R)$, where $\mathfrak{R}f = \hat{f}|_{S^{n-1}}$ is the sphere restriction operator. Note that $R(p,0)$ is equivalent to the global restriction estimate

$$\|\mathfrak{R}f\|_{L^p(S^{n-1})} \lesssim \|f\|_{L^p(\mathbf{R}^n)}.$$

The restriction conjecture[1] asserts that $R(p,0)$ holds for all $1 \leqslant p < 2n/(n+1)$. Localized restriction theorems such as the L^2-estimate $R(2,1/2)$ have been used before (see [1], [2]) as a stepping stone to global restriction theorems.

Although no formal equivalence has been proven between the two conjectures, they are widely believed to be at least heuristically equivalent. For example, the implication Restriction $\Rightarrow$ Bochner-Riesz is known for the parabolic analogue of the spherical problem (see Carbery [3]), or when the (p,p) restriction hypothesis is strengthened to a $(p,2)$-estimate (see, e.g., Fefferman [11], Christ [8], [6], [5], Tao [24]). Table 1 illustrates the close relationship between progress on the two conjectures.

In this paper we present a formal proof of the implication in the reverse direction, that the Bochner-Riesz conjecture implies the restriction conjecture, as an immediate consequence of the following two results.

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[1] There is a slight strengthening of this conjecture with imbalanced exponents, but we do not concern ourselves with that here.