Vol. 96, No. 2 DUKE MATHEMATICAL JOURNAL © 1999

ON THE SIZE OF DIFFERENTIAL MODULES

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In memory of Wei-Liang Chow, 1911–1995

§1. Introduction. We restrict our attention to systems of linear differential equations defined over $\mathbf{Q}(x)$, the field of rational functions in one variable with coefficients in the field $\mathbf{Q}$ of rational numbers. Let G be an $n \times n$ matrix with coefficients in $\mathbf{Q}(x)$, and consider the system

$$\frac{d\vec{y}}{dx} = \vec{y}G(x). \tag{1}$$

The differential module corresponding to such a system is said to be a G-module (or "of arithmetic type") if $\varrho(G)$, the inverse global radius of G, is finite. (For definitions, see [DGS, Chapter VII] or §2 here.)

If (1) is a G-module, then its size $\sigma(G)$ is also finite, and indeed by the theorem of E. Bombieri and Y. André [see [DGS, Chapter VII, Theorem 2.1])

$$\varrho(G) \leqslant \sigma(G) \leqslant \varrho(G) + n - 1.$$

(For a stronger form, see Theorem 1 here.) André [A, p. 76] erroneously cites the polylogarithm as proof that the Bombieri upper bound is the best possible. The size of that module is treated in §7 below.

All known G-modules "come from geometry," and it is a trivial consequence of [D1, §6] (restricting the argument to a generic disk rather than a singular one) that all modules coming from geometry are G-modules.

In the literature there are few precise computations of the size of modules. For modules of dimension 2 with at least one logarithmic singularity, we may deduce $\sigma - \varrho = 1$ from a theorem of André [A, p. 82] (see Theorem 4 here). André [A, p. 150] gave an asymptotic estimate for the size of the $(k+1)$-dimensional module corresponding to $(\log x)^k$. There is a statement in [A, p. 29] concerning the size of the series

$${}_kF_{k-1}(a_1, \dots, a_k; b_1, \dots, b_{k-1}, x),$$

but that statement is erroneous.

‡The editors of the *Duke Mathematical Journal* fondly remember Professor Bernard Dwork, who died in Princeton, New Jersey, on 9 May 1998.

Received 17 March 1997. Revision received 25 June 1997.
1991 *Mathematics Subject Classification.* 12H25, 33C70, 14D10