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BLOCH INVARIANTS OF HYPERBOLIC 3-MANIFOLDS

WALTER D. NEUMANN AND JUN YANG

1. Introduction. Let $M = \mathbb{H}^3/\Gamma$ be an oriented hyperbolic manifold of finite volume (so Γ is a torsion-free Kleinian group). It is known that M has a degree-one ideal triangulation by ideal simplices $\Delta_1, \dots, \Delta_n$ (see Section 2.1). Let $z_i \in \mathbb{C}$ be the parameter of the ideal simplex Δ_i for each i. These parameters define an element $\beta(M) = \sum_{i=1}^{n}[z_i]$ in the pre-Bloch group $\mathscr{P}(\mathbb{C})$ (as defined in Theorem 1.1 and in [14], for example).

THEOREM 1.1. *The above element $\beta(M)$ can be defined without reference to the ideal decomposition, so that it depends only on M. Moreover, it lies in the Bloch group $\mathscr{B}(\mathbb{C}) \subset \mathscr{P}(\mathbb{C})$.*

The independence of $\beta(M)$ on ideal triangulation holds even though our concept of degree-one ideal triangulation is rather more general than the usual ideal triangulation concepts.

We prove this theorem as follows. There is an exact sequence (mod 2-torsion) due to Bloch and Wigner (cf. [14])

$$0 \to \mu \to H_3(\mathrm{PGL}(2,\mathbb{C});\mathbb{Z}) \to \mathscr{B}(\mathbb{C}) \to 0,$$

where $\mu \subset \mathbb{C}^*$ is the group of roots of unity. If M is compact, then there is a "fundamental class" $[M] \in H_3(\mathrm{PGL}(2,\mathbb{C});\mathbb{Z})$, and we show that $\beta(M)$ is the image of $[M]$ in $\mathscr{B}(\mathbb{C})$. We do this by factoring through a certain relative homology group $H_3(\mathrm{PGL}(2,\mathbb{C}), \mathbb{CP}^1; \mathbb{Z})$ for which the relationship between $[M]$ and $\beta(M)$ is easier to see. (In fact, Dupont and Sah [14] show that this relative homology group maps isomorphically to $\mathscr{P}(\mathbb{C})$.) In the noncompact case we also find a fundamental class $[M]$ in this relative homology group that maps to $\beta(M) \in \mathscr{P}(\mathbb{C})$, thus proving that $\beta(M)$ is independent of triangulation. The fact that it lies in $\mathscr{B}(\mathbb{C})$ is the relation $\sum z_i \wedge (1 - z_i) = 0 \in \mathbb{C}^* \wedge \mathbb{C}^*$ on the simplex parameters z_i. For a more restrictive type of ideal triangulation than considered here, this relation has been attributed to Thurston (unpublished) by Gross [19] (according to [45]). It also follows easily from [32] (see also [27]). We give a cohomological proof here.

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