Vol. 94, No. 3 DUKE MATHEMATICAL JOURNAL © 1998

ON BOUNDS OF $N(\lambda)$ FOR A MAGNETIC SCHRÖDINGER OPERATOR

ZHONGWEI SHEN

To Professor Eugene Fabes

1. Introduction. In this paper we are concerned with the Schrödinger operator

$$H = H(\mathbf{a}, V) = -(\nabla - i\,\mathbf{a}(x))^2 + V(x) \quad \text{in } \mathbb{R}^n. \tag{1.1}$$

Assume that the magnetic potential $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, the electrical potential $V \geqslant 0$, and $V \in L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R})$. The quadratic form

$$h[\psi] = \int_{\mathbb{R}^n} |(\nabla - i\mathbf{a})\psi|^2 dx + \int_{\mathbb{R}^n} V|\psi|^2 dx \tag{1.2}$$

then generates a unique nonnegative selfadjoint operator in $L^2(\mathbb{R}^n, \mathbb{C})$, which we still denote by H.

Let $N(\lambda)$ denote the dimension of the spectral projection of H on $[0, \lambda)$ for $\lambda > 0$. The aim of this paper is to establish the upper and lower bounds for $N(\lambda)$ and the ground state energy E.

Given $\lambda > 0$, we divide $\mathbb{R}^n$ into a grid of mutually disjoint cubes $\{Q_\alpha\}$ of side $1/\sqrt{\lambda}$. For $1 < p \leqslant \infty$, let $\tilde{N}_p(\lambda)$ denote the number of cubes Q_α such that

$$C_p\left(\frac{1}{|Q_\alpha|}\int_{Q_\alpha} |\mathbf{B}|^p dx\right)^{1/p} + \left(\frac{1}{|Q_\alpha|}\int_{Q_\alpha} |V|^p dx\right)^{1/p} < \lambda, \tag{1.3}$$

where $\mathbf{B} = (b_{jk})_{1 \leqslant j,k \leqslant n}$, with

$$b_{jk} = \frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j}, \tag{1.4}$$

is the magnetic field generated by potential $\mathbf{a}$.

In the case where the magnetic potential is absent, that is, $H = -\Delta + V(x)$, it was proved by C. Fefferman and D. H. Phong (cf. [F, Theorem 3, p. 144]) that

Received 10 February 1997. Revision received 17 April 1997.
1991 *Mathematics Subject Classification.* 35P15, 35P20, 35Q40.
Author's work supported in part by National Science Foundation grant number DMS-9596266.