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ON THE VOLUMES OF CUSPED, COMPLEX HYPERBOLIC MANIFOLDS AND ORBIFOLDS

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The study of volumes of hyperbolic manifolds and orbifolds has been a fertile area of research for some time. In particular, the smallest-volume, cusped hyperbolic 3-orbifold and 3-manifold are found in [18] and [1], respectively. A key ingredient in this construction was the sphere-packing estimates of Böröczky [4]. In [12] Hersonsky and Paulin initiated a similar study for complex hyperbolic manifolds. In particular, in dimension 2 (we always give the complex dimension for complex hyperbolic manifolds and orbifolds) they use the complex hyperbolic Gauss-Bonnet formula to prove the following result.

Theorem A (Theorem 2.1 of [12]). *The smallest volume of a closed, complex hyperbolic 2-manifold is* $8\pi^2$.

In this paper we only consider cusped (that is noncompact, finite volume) complex hyperbolic manifolds and orbifolds. We prove the following theorem.

Theorem B. *The smallest volume of a cusped (and so of any) complex hyperbolic 2-manifold is* $8\pi^2/3$.

The ends of cusped complex hyperbolic manifolds and orbifolds necessarily have neighbourhoods corresponding to the quotient of a horoball by a cocompact group of Heisenberg isometries. The complex hyperbolic version of Shimizu's lemma (see Theorem 2.2 of [17] or Corollary 5.2 of [19]) gives a uniform estimate on the size of these horoballs and hence of the largest embedded cusp neighbourhood. When we have more than one end, we need to use canonical horoballs that are disjoint (see Proposition 5.7 of [12], which is a complex hyperbolic version of Proposition 3.3 of [11]). Because in complex hyperbolic space we do not have an analogue of Böröczky's result, these do not lead directly to estimates on the volume of the whole manifold or orbifold. Every cocompact $(2n-1)$-dimensional group of Heisenberg isometries contains a Heisenberg lattice as a subgroup of finite index. For the fundamental groups of Heisenberg $(2n-1)$-manifolds, this index is at most I_n, and for $(2n-1)$-orbifolds at most J_n. It is known that $I_n \leqslant 2(6\pi)^{(2n-1)(n-1)}$ (Theorem 1.5(ii) of [5]) and $I_2 = J_2 = 6$ (Proposition 5.8 of [12]; their result is only stated for manifolds, but the proof works for orbifolds as well). Using this Hersonsky and Paulin proved the following result.

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