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GENUS 1 ENUMERATIVE INVARIANTS IN $\mathbb{P}^n$ WITH FIXED j INVARIANT

ELENY IONEL

0. Introduction. A classical problem in enumerative algebraic geometry is to compute the number of degree d, genus g holomorphic curves in $\mathbb{P}^n$ that pass through a certain number of constraints (points, lines, etc.).

Let σ_d denote the number of degree d rational curves ($g = 0$) through appropriate constraints. For example, $\sigma_1(pt, pt) = 1$ (since two points determine a line). The first nontrivial cases were computed around 1875 when Schubert, Halphen, Chasles, et al. found σ_2 for $\mathbb{P}^2$ and $\mathbb{P}^3$. Later, more low-degree examples were computed in $\mathbb{P}^2$ and $\mathbb{P}^3$, but the progress was slow. Then in 1993 Kontsevich [KM] predicted, based on ideas of Witten, that the number σ_d of degree d rational curves in $\mathbb{P}^2$ through $3d-1$ points satisfies the recursive relation

$$\sigma_d = \sum_{d_1+d_2=d} \left[\binom{3d-1}{3d_1-1} d_1^2 d_2^2 - \binom{3d-1}{3d_1-2} d_1^3 d_2 \right] \sigma_{d_1} \sigma_{d_2},$$

where $d_i \neq 0$ and $\sigma_1 = 1$. Ruan-Tian [RT] extended these formulas for σ_d in any $\mathbb{P}^n$.

When genus $g = 1$, the classical problem splits into two totally different problems: one can count (i) elliptic curves with a fixed complex structure or (ii) elliptic curves with unspecified complex structure (each satisfying the appropriate number of constraints). This paper gives recursive formulas which completely solve the first of these.

Thus our goal is to compute the number τ_d of degree d elliptic curves in $\mathbb{P}^n$ with fixed j invariant. Classically, the progress on this problem has been even slower than on the genus 1 case. Recently, Pandharipande [Pan] found recursive formulas for τ_d for the 2-dimensional projective space $\mathbb{P}^2$ using the Kontsevich moduli space of stable curves.

We will approach the problem from a different direction, using analysis. Our approach is based on the ideas introduced by Gromov to study symplectic topology. If (Σ, j) is a fixed Riemann surface, let

$$\{f : \Sigma \to \mathbb{P}^n | \bar{\partial}_J f = 0,\ [f] = d \cdot l \in H_2(\mathbb{P}^n, \mathbb{Z})\} / \mathrm{Aut}(\Sigma, j)$$

be the moduli space of degree d holomorphic maps $f : \Sigma \to \mathbb{P}^n$, modulo the auto-

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