Vol. 93, No. 3 DUKE MATHEMATICAL JOURNAL © 1998

# ROOT NUMBERS AND ALGEBRAIC POINTS ON ELLIPTIC SURFACES WITH ELLIPTIC BASE

GREGORY R. GRANT AND ELISABETTA MANDUCHI

**Introduction.** Let $X$ be an algebraic variety defined over a number field $F$. Say $X$ has property $(Z)$ if there exists a finite extension $L$ of $F$ such that $X(L)$ is Zariski dense in $X$. Lang conjectured [6] that if $X$ is of general type, then $X$ does not have property $(Z)$. One would then like to know in what generality property $(Z)$ does hold. From Lang's conjecture it follows that if $X$ dominates a variety of general type, then $X$ does not have property $(Z)$. Until recently (1996), there was no known example of a variety that does not dominate a variety of general type and yet does not have property $(Z)$ (see [1], [2]), and there is scant evidence available to study this situation. In [3], the authors provide evidence, based on a natural generalization of Birch and Swinnerton-Dyer, that if $\mathscr{E}$ is an elliptic surface (as defined in [10], hence with a section) with base $\mathbb{P}^1$ and with nonconstant $j$-invariant, then $\mathscr{E}$ has property $(Z)$. In this paper, we provide the same kind of evidence for elliptic surfaces with base an *elliptic curve* and with nonconstant $j$-invariant. By Falting's theorem, this settles entirely the question for elliptic surfaces with nonconstant $j$-invariant.

We use the same notation as in [3]. Namely, if $L$ is a finite Galois extension of a number field $K$, $\tau$ is a complex representation of $\mathrm{Gal}(L/K)$ with real-valued character, and $E$ is an elliptic curve defined over $K$, then we denote by $W(E/K,\tau)$ the root number associated to $E$ and $\tau$, which has an intrinsic definition as a product of local factors (see [8]). The conjectures of Birch and Swinnerton-Dyer and Deligne and Gross (see [7]) imply that

$$W(E_P/K,\tau) = (-1)^{\langle \rho_{E_P},\tau\rangle}, \tag{1}$$

where $\rho_E$ is the natural complex representation of $\mathrm{Gal}(L/K)$ on $E(L)\otimes_{\mathbb{Z}}\mathbb{C}$.

We prove the following result on root numbers, which is independent of any conjectures.

THEOREM. *Let $\mathscr{E}$ be an elliptic surface defined over a number field $F$, with base an elliptic curve $C$ and with nonconstant $j$-invariant. Then there exist a finite extension $K$ of $F$, a finite Galois extension $L$ of $K$, and a complex irreducible representation $\tau$ of* $\mathrm{Gal}(L/K)$*, with real-valued character, such that, if $E_P$ denotes the fiber of $\mathscr{E}$ over $P$, then the set of $P$'s in $C(K)$ for which the root number*

Received 2 January 1997. Revision received 11 March 1997.
1991 *Mathematics Subject Classification.* Primary 11; Secondary 14.