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BONNET PAIRS AND ISOTHERMIC SURFACES

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1. Introduction. A classical question in surface theory is which data are sufficient to describe a surface in space up to rigid motions. Bonnet suggested that the mean curvature and metric should suffice to determine the surface generically. The local theory (without umbilic points) was developed by Bonnet [3], Cartan [5], and Chern [6], who showed the existence of various 1-parameter families of Bonnet surfaces, that is, surfaces with the same induced metric and mean curvature. A comprehensive study of this problem and its relationship to the Painlevé equations has recently been completed by Bobenko and Eitner [1]. On the other hand, Lawson and Tribuzy [9] have shown that for embedded compact surfaces, there are at most two surfaces to a given metric and mean curvature. Moreover, uniqueness can be established under various global assumptions (see [8]). Up to now, it is unknown whether such compact Bonnet pairs exist and, if they do, how to construct them.

In this note we describe *all* Bonnet pairs on a simply connected domain, including surfaces with umbilic points. Our main intent is to apply what we call a *quaternionic function theory* to a concrete problem in differential geometry. In the first section, we develop the necessary formulas and explain how usual Riemann surface theory can be viewed as a special case of our extended function theory. The ideas are simple: conformal immersions into quaternions or imaginary quaternions take the place of chart maps for a Riemann surface. Starting from a reference immersion, we construct all conformal immersions of a given Riemann surface (up to translational periods) by *spin transformations* (see Definition 7). With this viewpoint in mind, we discuss in the second section how to construct all Bonnet pairs on a simply connected domain from isothermic surfaces, and vice versa. Isothermic surfaces are solutions to a certain soliton equation (see [4]); thus a simple dimension count tells us that we obtain Bonnet pairs that are not part of any of the families. The correspondence between Bonnet pairs and isothermic surfaces is explicit, and to each isothermic surface we obtain a 4-parameter family of Bonnet pairs. An indication that such an explicit relationship might exist was first given by Cartan, who observed that the general (local) Bonnet pair depends on four arbitrary functions of one variable, which is the same generality as the general (local) isothermic surface.

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