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NONCOMMUTATIVE DEFORMATIONS OF KLEINIAN SINGULARITIES

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0. Introduction. Let K be an algebraically closed field of characteristic zero and let Γ be a nontrivial finite subgroup of $\mathrm{SL}_2(K)$. Deformations of the Kleinian singularity K^2/Γ have been studied for some time, especially the semiuniversal deformation (see, for example, Slodowy [34]). In this paper we define and study a family of deformations of its coordinate ring $K[x,y]^\Gamma$. Our rings, denoted $\mathcal{O}^\lambda$ for $\lambda \in Z(K\Gamma)$, are, in general, noncommutative, even simple rings. For example, $\mathcal{O}^1$ is the fixed ring of Γ on the Weyl algebra $K\langle x, y | xy - yx = 1\rangle$. However, if the trace of λ on the regular representation $K\Gamma$ is zero, then $\mathcal{O}^\lambda$ is commutative, and (in the case $K = \mathbb{C}$) the complex analytic spaces arising from the schemes $\operatorname{Spec} \mathcal{O}^\lambda$ are precisely the fibres of the semiuniversal deformation.

In the special case when Γ is cyclic, a family of noncommutative deformations was studied by Hodges in [15] (see also [18], [36], and [2]). These rings are defined by generators and relations and were originally studied as generalisations of the primitive factors of the enveloping algebra of $\mathfrak{sl}_2$. Hodges noted that families of deformations should exist for general Γ, but it would be impractical to find them using his methods.

In retrospect, it is easy to write down deformations of $K[x,y]^\Gamma$ for any Γ. The group acts naturally on the ring $K\langle x, y\rangle$ of noncommuting polynomials, so one can form the skew group algebra $K\langle x, y\rangle * \Gamma$. For $\lambda \in Z(K\Gamma)$, let $\mathcal{S}^\lambda$ be the quotient $(K\langle x, y\rangle * \Gamma)/(xy - yx - \lambda)$. We then define $\mathcal{O}^\lambda = e\mathcal{S}^\lambda e$, where $e \in K\Gamma$ is the average of the group elements.

The ring $\mathcal{O}^\lambda$ is naturally filtered, and it is easy to see that the associated graded algebra is isomorphic to $K[x,y]^\Gamma$. It follows that $\mathcal{O}^\lambda$ is a Noetherian domain, finitely generated as a K-algebra, of Gelfand-Kirillov (GK) dimension 2, it is a maximal order in its quotient ring, and it is Auslander-Gorenstein and Cohen-Macaulay. Other properties of $\mathcal{O}^\lambda$ depend on the incidence of λ with hyperplanes in $Z(K\Gamma)$ perpendicular to elements of the affine root system corresponding to the McKay graph [28] of Γ. To investigate this, we relate $\mathcal{O}^\lambda$ to a family of deformations of the preprojective algebra because these deformations can be studied using a new type of reflection functor. We begin by defining the McKay graph and the deformed preprojective algebra.

Let $N_0, \dots, N_n$ be the irreducible representations of Γ with N_0 trivial, and let V be the natural 2-dimensional representation of Γ. The *McKay graph* of Γ is

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