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COMMUTATORS OF FREE RANDOM VARIABLES

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1. Introduction and presentation of the results. In this paper, we show how the combinatorial description of freeness can be used to determine the distribution of the commutator of two free random variables.

The concept of freeness was introduced by Voiculescu [14] as a tool for studying free products of operator algebras. It soon became clear that it is a promising point of view to consider freeness as a noncommutative analogue of the classical probabilistic notion of independence, and this led to the development of a free probability theory (see the monograph [18], or the recent survey in [17]).

Let a and b be free random variables. Two basic problems in free probability theory, both solved by Voiculescu, consist in the study of $a+b$ and ab. The distributions of these new random variables are given by the additive ($\boxplus$) and, respectively, multiplicative ($\boxtimes$) free convolution of the distributions of a and b. Voiculescu [15], [16] provided with the R- and S-transform two efficient analytical tools for dealing with $\boxplus$ and $\boxtimes$.

To be more precise, let us recall that if $(\mathscr{A}, \varphi)$ is a *noncommutative probability space*—that is, $\mathscr{A}$ is a unital algebra, endowed with a linear functional $\varphi : \mathscr{A} \to \mathbf{C}$ such that $\varphi(1) = 1$—then the *distribution* μ_a of an element $a \in \mathscr{A}$ is the linear functional on $\mathbf{C}[X]$ defined by $\mu_a(f) = \varphi(f(a))$ for all $f \in \mathbf{C}[X]$. For every such distribution $\mu : \mathbf{C}[X] \to \mathbf{C}$ (μ linear, $\mu(1) = 1$), one defines its R-transform $R(\mu)$ and its S-transform $S(\mu)$ as special formal power series in an indeterminate z in such a way that, for a and b free in some $(\mathscr{A}, \varphi)$, we have the formulas

$$R(\mu_{a+b}) = R(\mu_a) + R(\mu_b) \tag{1.1}$$

and

$$S(\mu_{ab}) = S(\mu_a) \cdot S(\mu_b), \qquad \text{if } \varphi(a) \neq 0 \neq \varphi(b). \tag{1.2}$$

In the case of selfadjoint operators, where the distributions of the random variables can be viewed as probability measures, the R- and the S-transforms are related with the corresponding Cauchy transforms. (For the definition of freeness, see Definition 2.1.4 below—or, for more details, Section 2.5 of [18]. For the

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