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CHORD DIAGRAM INVARIANTS OF TANGLES AND GRAPHS

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The notion of a chord diagram emerged from Vassiliev's work [V1], [V2] (see also Gusarov [Gu1], [Gu2] and Bar-Natan [BN1], [BN2]). Slightly later, Kontsevich [Kn] defined an invariant of classical knots taking values in the algebra generated by formal linear combinations of chord diagrams modulo the four-term relation. This knot invariant establishes an isomorphism of a projective limit of algebras generated by the Vassiliev equivalence classes of knots onto the algebra of chord diagrams.

Kontsevich originally defined his invariant of knots via a multiple integral given by an explicit but complicated analytic expression. This expression, however beautiful, does not reveal the combinatorial nature of the invariant. (A similar situation would occur if the linking number of knots were introduced via the Gauss integral formula without a combinatorial calculation.) A combinatorial reformulation of the Kontsevich integral appeared in the works of Bar-Natan [BN3], Cartier [C], Le and Murakami [LM1], Piunikhin [P] (see also [Ka, Chapter XX]). On the algebraic side, this reformulation uses the notions of braided and infinitesimal symmetric categories as well as the notion of an associator introduced by Drinfeld [D3] in his study of quasitriangular quasi-Hopf algebras. On the geometric side, one uses categories of tangles, as introduced by Yetter and Turaev (see [T]). Note also that a counterpart of the Kontsevich knot invariant in the theory of braids was discovered earlier by Kohno [Ko], who considered an algebraic version of chord diagrams.

In this paper we clarify the relationship between tangles and chord diagrams. It is formulated in terms of categories whose sets of morphisms are spanned by tangles and chord diagrams, respectively. More precisely, we fix a commutative ring R and consider categories $\mathscr{T}(R)$ and $\mathscr{A}(R)$ whose morphisms are formal linear combinations of framed oriented tangles and chord diagrams with coefficients in R (cf. Section 2). The set of morphisms in $\mathscr{T}(R)$ has a canonical filtration given by the powers of an ideal I, which we call the augmentation ideal. Functions on morphisms in $\mathscr{T}(R)$ vanishing on I^{m+1} are exactly the Vassiliev invariants of degree $\leqslant m$ for framed oriented tangles. Completing $\mathscr{T}(R)$ at the ideal I, we obtain the prounipotent completion $\hat{\mathscr{T}}(R) = \varprojlim_m \mathscr{T}(R)/I^{m+1}$. Our main result (Corollary 2.5) states that if R contains the field $\mathbf{Q}$ of rational numbers, then $\hat{\mathscr{T}}(R)$ is isomorphic to a suitable completion $\hat{\mathscr{A}}(R)$ of the cat-

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