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HYPERELLIPTIC RIEMANN SURFACES OF INFINITE GENUS AND SOLUTIONS OF THE KDV EQUATION

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1. Introduction. In this paper, we construct new classes of quasi-periodic solutions of the Korteweg-de Vries equation

$$u_t = 6uu_x - u_{xxx}, \quad u|_{t=0} = q, \tag{1.1}$$

where q is a function of $x \in \mathbb{R}$. To explain our main result, we recall some basic facts about the Cauchy problem for the KdV equation with periodic initial conditions. The investigation of this problem was started independently by Novikov [17], Lax [10], Matveev [12], and McKean and van Moerbeke [13], and it is now well known that for initial conditions q that are smooth and periodic in $x \in \mathbb{R}$, the integration of (1.1) is closely related to the inverse problem for the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + q(x) \tag{1.2}$$

with the periodic potential q. An important part in the study of the periodic (or, more generally, quasi-periodic) problem for the KdV equation is the investigation of the class of potentials q for which the Schrödinger operator (1.2) has only finitely many gaps (zones of instability) in the spectrum on the whole real line. The associated spectral surface $\mathscr{M}$ is a hyperelliptic Riemann surface of genus g (=number of gaps), and the theory of abelian functions can be used to reconstruct the potential from the spectrum of the periodic Schrödinger operator and to solve the KdV equation with these initial conditions. More precisely, for finite-zone periodic potentials, the solution of the KdV equation can be expressed explicitly by the Its-Matveev formula in terms of the Riemann theta functions associated to the corresponding spectral surface [8], [5], and [4].

This approach can be generalized, eliminating all reference to the Schrödinger operator. The boundaries of the finitely many gaps can be prescribed arbitrarily, and all results extend to this more general setting. In particular, the solutions of the KdV equation are given by an extension of the Its-Matveev formula, which we describe now. Let $\mathscr{M}$ be the hyperelliptic Riemann surface associated with

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