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THE COHOMOLOGY OF A COXETER GROUP WITH GROUP RING COEFFICIENTS

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Introduction. Let (W, S) be a Coxeter system with S finite (that is, W is a Coxeter group and S is a distinguished set of involutions which generate W, as in [B, p. 11.]). Associated to (W, S) there is a certain contractible simplicial complex Σ, defined below, on which W acts properly and cocompactly. In this paper we compute the cohomology with compact supports of Σ (that is, we compute the "cohomology at infinity" of Σ). As consequences, given a torsion-free subgroup Γ of finite index in W, we get a formula for the cohomology of Γ with group ring coefficients, as well as a simple necessary and sufficient condition for Γ to be a Poincaré duality group.

Given a subset T of S denote by W_T the subgroup generated by T. (If T is the empty set, then W_T is the trivial subgroup.) Denote by $\mathscr{S}^f$ the set of those subsets T of S such that W_T is finite; $\mathscr{S}^f$ is partially ordered by inclusion. Let $W\mathscr{S}^f$ denote the set of all cosets of the form wW_T, with $w \in W$ and $T \in \mathscr{S}^f$. $W\mathscr{S}^f$ is also partially ordered by inclusion.

The simplicial complex Σ is defined to be the geometric realization of the poset $W\mathscr{S}^f$. The geometric realization of the poset $\mathscr{S}^f$ will be denoted by K.

For each s in S, let $\mathscr{S}^f_{\geqslant\{s\}}$ be the subposet consisting of those $T \in \mathscr{S}^f$ such that $s \in T$ and let K_s be its geometric realization. So, K_s is a subcomplex of K. (K is called a *chamber* of Σ and K_s is a *mirror* of K.) For any nonempty subset T of S, set

$$K^T = \bigcup_{s \in T} K_s.$$

K is a contractible finite complex; it is homeomorphic to the cone on K^S.

For each $w \in W$, set

$$S(w) = \{s \in S | \ell(ws) < \ell(w)\}$$

$$T(w) = S - S(w),$$

where $\ell(w)$ is the minimum length of word in S which represents w. Thus, $S(w)$ is the set of elements of S in which a word of minimum length for w can end.

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