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# THE DIOPHANTINE EQUATION $Ax^p + By^q = Cz^r$

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**1. Introduction.** Let $p, q, r \in \mathbf{Z}_{\geqslant 2}$ and $A, B, C \in \mathbf{Z}$ with $ABC \neq 0$. Consider the diophantine equation

$$Ax^p + By^q + Cz^r = 0, \qquad \gcd(x, y, z) = 1, \quad xyz \neq 0, \tag{F}$$

in the unknowns $x, y, z \in \mathbf{Z}$.

It was proved by Darmon and Granville in 1993 [DG] that if $1/p + 1/q + 1/r < 1$, then (F) has finitely many solutions. We call this the *hyperbolic case*. As a curiosity, consider the case $A = B = -C = 1$. The only solutions known until now are, up to permutations and sign changes,

$$1^k + 2^3 = 3^2\ (k > 6), \qquad 13^2 + 7^3 = 2^9, \qquad 2^7 + 17^3 = 71^2,$$

$$2^5 + 7^2 = 3^4, \qquad 3^5 + 11^4 = 122^2, \qquad 17^7 + 76271^3 = 21063928^2,$$

$$1414^3 + 2213459^2 = 65^7, \qquad 33^8 + 1549034^2 = 15613^3,$$

$$43^8 + 96222^3 = 30042907^2, \qquad 9262^3 + 15312283^2 = 113^7.$$

The smaller solutions have been known for a long time; the larger ones were found by a computer search performed on Fermat day at Utrecht in November 1993, a day devoted to Wiles's work on Fermat's last theorem. Notice that in each solution an exponent 2 occurs. This leads to the following question raised by Tijdeman and Zagier.

*Question 1.1.* Suppose $p, q, r \geqslant 3$. Do there exist solutions to $x^p + y^q = z^r$ in $x, y, z \in \mathbf{Z}$ with $xyz \neq 0$ and $\gcd(x, y, z) = 1$?

For further reading concerning the hyperbolic case, see [DM].

When $1/p + 1/q + 1/r = 1$ (the *Euclidean case*), we can list the possible sets $\{p, q, r\}$ by $\{3, 3, 3\}, \{2, 4, 4\}, \{2, 3, 6\}$. In this case we find ourselves looking at the problem of rational points on elliptic curves with $j$-values zero and 1728. The study of rational points on elliptic curves has become a vast field of interest in the last ten years, to which we have nothing to add in this paper.

Finally, we have the so-called *spherical case* $1/p + 1/q + 1/r > 1$. The possible sets $\{p, q, r\}$ are $\{2, 2, k\}$ with $k \geqslant 2$ and $\{2, 3, 3\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$. Apart from

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