Vol. 91, No. 1 DUKE MATHEMATICAL JOURNAL

THE HORIZONTAL DISTRIBUTION OF ZEROS OF $\zeta'(s)$

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1. Introduction. Throughout this paper, $\rho = \beta + i\gamma$ denotes a typical complex zero of $\zeta(s)$, and $\rho_1 = \beta_1 + i\gamma_1$ denotes a generic complex zero of $\zeta'(s)$. Further let us mention at the outset that all implied constants below are absolute.

In this paper, we are concerned with the distribution of the β_1's: that is, the horizontal distribution of the zeros of $\zeta'(s)$. The distribution of zeros of $\zeta'(s)$ is intimately related to the distribution of zeros of $\zeta(s)$. An example of this connection is an old result of A. Speiser [Sp], which says that the Riemann hypothesis (RH) is equivalent to the nonexistence of nonreal zeros of $\zeta'(s)$ in the region to the left of the "critical" $\sigma = \Re s = 1/2$ line. In other words, $\beta = 1/2$ is equivalent to $\inf \beta_1 \geqslant 1/2$. This relation has been quantified by N. Levinson and H. L. Montgomery [LM], who showed that

$$\#\{\rho : \beta < 1/2,\ 0 < \gamma \leqslant T\} = \#\{\rho_1 : \beta_1 < 1/2,\ 0 < \gamma_1 \leqslant T\} + O(\log T). \tag{1.1}$$

The above relation leads us to believe that unlike the zeros of $\zeta(s)$, which are (by the functional equation) located symmetrically about the line $\sigma = 1/2$, the zeros of $\zeta'(s)$ have an asymmetrical horizontal distribution. This sentiment is confirmed by the following result of Levinson and Montgomery:

$$\sum_{0<\gamma_1\leqslant T} (\beta_1 - 1/2) \sim \frac{T}{2\pi}\log\log T. \tag{1.2}$$

Since, by a result of B. C. Berndt [B], the number of zeros of $\zeta'(s)$ with ordinate less than T is

$$N_1(T) := \sum_{0\leqslant\gamma_1\leqslant T} 1 = \frac{T}{2\pi}\log\frac{T}{4\pi e} + O(\log T), \tag{1.3}$$

it follows from (1.2) that the average value of β_1 is $1/2 + \log\log T/\log T$. Recall that $N(T)$ denotes the number of zeros of $\zeta(s)$ with ordinate γ satisfying $0 < \gamma \leqslant T$. It is well known (see [T]) that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T).$$

Received 22 January 1996. Revision received 25 October 1996.