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THE NODAL LINE OF THE SECOND EIGENFUNCTION OF THE LAPLACIAN IN $\mathbb{R}^2$ CAN BE CLOSED

M. HOFFMANN-OSTENHOF, T. HOFFMANN-OSTENHOF, AND N. NADIRASHVILI

Let D be a bounded domain in $\mathbb{R}^2$. Consider the corresponding Dirichlet problem

$$-\Delta u_i = \lambda_i u_i, \quad i = 1, 2 \ldots, \tag{1}$$

with $u_i \in W_0^{1,2}(D)$ (the closure of $C_0^\infty(D)$ in the $W^{1,2}$-norm [GT]) and with eigenvalues $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$.

We see that u_1 has one sign but the nodal set of u_2, $\mathcal{N}(u_2) = \{x \in D : u_2(x) = 0\}$ is nonempty. In fact, according to Courant's nodal theorem, there are domains D_+ and D_- such that $\bar{D} = \overline{D_+ \cup D_-}$ with $u_2 > 0$ in D_+, $u_2 < 0$ in D_-, and $u_2 = 0$ in ∂D_+ and ∂D_-.

In 1967 Payne [P] conjectured that u_2 cannot have a closed nodal line in D. In 1982 Yau [Y] asked the same question for convex domains in $\mathbb{R}^2$. Melas [M] recently has settled the convex case for the C^∞-boundary and this was extended to the general boundary by Alessandrini [A]. Also, for convex D Jerison [J] and Grieser and Jerison [GJ] obtained interesting results on the location of $\mathcal{N}(u_2)$.

In this note we construct a nonconvex, not simply connected domain for which the second eigenfunction has a closed nodal line.

We first describe the domain. We use polar coordinates, $r = |x|$, $x_1 = r\cos\omega$, and $x_2 = r\sin\omega$, $-\pi \leqslant \omega \leqslant \pi$.

Let $0 < R_1 < R_2$, $B_{R_i} = \{x \in \mathbb{R}^2 : r < R_i\}$, $i = 1, 2$, and the annulus $M_{R_1,R_2} = B_{R_2} \backslash \overline{B_{R_1}}$. We shall construct our domain by opening passages between M_{R_1,R_2} and B_{R_1}. First we pick R_1 and R_2 such that

$$\lambda_1(B_{R_1}) < \lambda_1(M_{R_1,R_2}) < \lambda_2(B_{R_1}), \tag{2}$$

where the $\lambda_i(\cdot)$ denote the corresponding Dirichlet eigenvalues. In particular, we then have

$$\begin{aligned} &\lambda_1(B_{R_1} \cup M_{R_1,R_2}) = \lambda_1(B_{R_1}) \quad \text{and} \\ &\lambda_2(B_{R_1} \cup M_{R_1,R_2}) = \lambda_1(M_{R_1,R_2}). \end{aligned} \tag{3}$$

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