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CYCLES OF QUADRATIC POLYNOMIALS AND RATIONAL POINTS ON A GENUS-2 CURVE

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1. Introduction. Let $g(z) \in \mathbb{Q}(z)$ be a rational function of degree $d \geqslant 2$. We consider g as a map on $\mathbb{P}^1(\mathbb{C})$. If $x \in \mathbb{P}^1(\mathbb{C})$ and the sequence

$$x, g(x), g(g(x)), \ldots, g^{\circ n}(x), \ldots$$

is eventually periodic, then x is called a *preperiodic point* for g. If, furthermore, $g^{\circ n}(x) = x$, then x is called a *periodic point* of g of period n, and its orbit

$$\{x, g(x), g(g(x)), \ldots, g^{\circ(n-1)}(x)\}$$

is called an *n-cycle* if x does not actually have smaller period. Northcott [31] proved in 1950 that for fixed g, there are only finitely many preperiodic points in $\mathbb{P}^1(\mathbb{Q})$. Moreover, these can be computed effectively given g. This theorem also holds over any fixed number field, and also for morphisms of $\mathbb{P}^n$ of degree at least 2. Since then, the theorem (in varying degrees of generality) has been rediscovered by many authors [30], [20], [2].

It is much more difficult to obtain *uniform* results for rational functions of a given degree. Morton and Silverman [28] have proposed the following conjecture.

Conjecture 1. *Let $K/\mathbb{Q}$ be a number field of degree D, and let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be a morphism of degree $d \geqslant 2$ defined over K. The number of K-rational preperiodic points of ϕ can be bounded in terms of D, n, and d only.*

Silverman, in talks on the subject, has pointed out that even the case $n = 1$ and $d = 4$ is strong enough to imply the recently proved strong uniform boundedness conjecture for torsion of elliptic curves (see [23]); namely, that for any D there exists $C > 0$ such that for any elliptic curve E over a number field K of degree D over $\mathbb{Q}$, $\# E(K)_{\text{tors}} < C$. This is because torsion points of elliptic curves are exactly the preperiodic points of the multiplication-by-2 map, and their x-coordinates are preperiodic points for the degree-4 rational map that gives

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