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THE STABLE RANK OF SOME FREE PRODUCT C^*-ALGEBRAS

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Introduction. We prove that the reduced group C^*-algebra $C^*_{\text{red}}(G)$ has stable rank 1 if G is a discrete group arising as a free product $G_1 * G_2$ where $|G_1| \geqslant 2$ and $|G_2| \geqslant 3$. Recall that a unital C^*-algebra A is said to have stable rank 1 if the group of invertible elements in A is a norm dense subset of A. It is an open problem[1] if every finite, simple C^*-algebra has stable rank 1. Our result follows from a more general result where it is proved that if $(\mathfrak{A}, \tau)$ is the reduced free product of a family (A_i, τ_i), $i \in I$, of unital C^*-algebras A_i with normalized faithful traces τ_i, and if the family satisfies the Avitzour condition (that is, the traces, τ_i, are not too lumpy in a specific sense), then $\mathfrak{A}$ has stable rank 1.

The notion of stable rank was introduced by M. Rieffel in [11] with the purpose of establishing what one might call nonstable K-theory results for certain concrete C^*-algebras, most notably the irrational rotation C^*-algebras. On a more speculative note, stable rank (which associates a number in $\{1, 2, 3, \ldots\} \cup \{\infty\}$ to every C^*-algebra) should measure the (noncommutative) dimension of the C^*-algebra, with stable rank equal to one corresponding to dimension 0 or 1. (It has later turned out that different definitions of dimensions, which agree in the "commutative" case, generalize to dimension concepts for C^*-algebras which do not agree.)

Some of the nonstable K-theory results, obtained in [11] and [13], for unital C^*-algebras A of stable rank 1 are as follows. The three relations on projections in A (or in a matrix algebra over A), Murray-von Neumann equivalence, unitary equivalence, and homotopy equivalence, are the same. Moreover, if p, q are projections in A (or in a matrix algebra over A) such that $[p]_0 = [q]_0$ in $K_0(A)$, then p and q are equivalent with respect to either of the three relations mentioned above. Also, the natural group homomorphism

$$\mathrm{U}(A)/\mathrm{U}_0(A) \to K_1(A),$$

where $\mathrm{U}(A)$ is the group of unitary elements in A and $\mathrm{U}_0(A)$ its connected component containing the unit of A, is an isomorphism.

Another property of C^*-algebras of stable rank 1 can be found in [7]. It is observed in that paper that if A is a C^*-algebra of stable rank 1, then each

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[1] After this paper was written, Jesper Villadsen [15] found a stably finite, simple C^*-algebra with stable rank greater than 1.