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GLOBAL VERSUS LOCAL ASYMPTOTIC THEORIES OF FINITE-DIMENSIONAL NORMED SPACES

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1. Introduction. This paper is devoted to the comparison between global properties and local properties of symmetric convex bodies of high dimension. By global properties, we refer to properties of the original body in question and its images under linear transformations. The local properties pertain to the structure of lower-dimensional sections and projections of the body, that is, to the linear structure of a normed space in the spirit of functional analysis. In both theories, we are interested in the asymptotic behavior, as the dimension grows to infinity, of the relevant quantities. Also, as is common in such questions, the approach we consider does not yield exact isometric results but rather falls into the isomorphic (and for some results, the almost isometric) category.

Unexpectedly, it appears that there is an exact parallelism between the two theories: the global (geometric) asymptotic theory and the local theory. We will demonstrate that several well-known facts of local type have corresponding equivalent geometric results and vice versa. Let us describe, as an example, a well-known classical local theory fact—Dvoretzky's theorem.

A well-known version of Dvoretzky's theorem states: *Let $X = (\mathbb{R}^n, \|\cdot\|)$ be an n-dimensional normed space. Let $|\cdot|$ denote the canonical Euclidean norm on $\mathbb{R}^n$. Put $M = \int_{S^{n-1}} \|x\| d\mu(x)$ (where μ is the normalized Haar measure on the Euclidean sphere), and assume $\|x\| \leqslant b|x|$ for all $x \in \mathbb{R}^n$. Let $\varepsilon > 0$; then for some absolute constant c and for $k = [c\varepsilon^2 n(M/b)^2]$, there exists a subspace $E \in G_{n,k}$ (the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$) for which*

$$\frac{M}{1+\varepsilon}\|x\| \leqslant |x| \leqslant (1+\varepsilon)M\|x\| \qquad \textit{for all } x \in E\ . \tag{1}$$

Moreover, the subset of $G_{n,k}$ satisfying this has measure tending fast to 1 as $n \to \infty$. (Here, the measure is the normalized Haar measure on $G_{n,k}$.) (Compare [Mil], [FLM], [MS], [Pi], [G], [Sc].)

We will refer to this statement, and others similar to it pertaining to the behavior of a section (subspace) of convex bodies (normed spaces), as "local statements." In particular, the above is a local form of Dvoretzky's theorem.

In [BLM] (see also [Schm]), it was proved (although not stated–we shall return to this in the proof of Theorem 2.2 below) that, under the same condi-

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