GENERALIZED LAPLACE INVARIANTS AND THE METHOD OF DARBOUX

MARTIN JURÁŠ AND IAN M. ANDERSON

§1. Introduction. The methods of Laplace, Ampère, and Darboux for the exact integration of second-order, hyperbolic equations in the plane have been studied in a number of recent papers (see [1], [2], [5], [7], [15], and [17]). The current activity in this classical subject is motivated, in part, by a general resurgence of interest in geometric methods for the study of differential equations and also, in part, by a desire to classify various types of completely integrable partial differential equations [14]. In [1], it was established that if a hyperbolic equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \tag{1.1}$$

is Darboux integrable, then the associated sequence of *generalized Laplace invariants* is finite. In this paper we prove, conversely, that the finiteness of the sequence of generalized Laplace invariants for (1.1) insures that this equation is integrable by the Darboux method. Our result generalizes to the case of fully nonlinear equations the well-known result that a linear equation

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0 \tag{1.2}$$

is integrable by the method of Darboux, or equivalently, by the method of Laplace, if and only if the sequence of classical Laplace invariants is finite. Our result also generalizes the recent work of Sokolov and Ziber [15], who consider the special class of Monge-Ampère equations

$$u_{xy} = f(x, y, u, u_x, u_y).$$

They prove, by a rather ingenious method, that the finiteness of the sequence of generalized Laplace invariants for this equation is sufficient for the successful application of the method of Darboux.

Our main result can be stated more precisely as follows. We first recall, by way of motivation, that the first terms in the sequence of Laplace invariants for the linear equation (1.2) are

$$h_0 = \frac{\partial a}{\partial x} + ab - c \qquad \text{and} \qquad k_0 = \frac{\partial b}{\partial y} + ab - c.$$

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