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WEYL-HEISENBERG FRAMES AND RIESZ BASES IN $L_2(\mathbb{R}^d)$

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1. Introduction

1.1. Frames, Riesz bases, and their dual systems. The present paper is the second in a series of three, all devoted to the study of *shift-invariant frames* and *shift-invariant stable (= Riesz) bases* for $H := L_2(\mathbb{R}^d)$, $d \geqslant 1$, or a subspace of it. In the first paper [RS1] we studied such bases under the mere assumption that the basis set can be written as a collection of *shifts* (namely, integer translates) of a set of *generators* Φ. The present paper analyzes Weyl-Heisenberg (WH, known also as Gaborian) frames and stable bases. Aside from specializing the general methods and results of [RS1] to this important case, we exploit here the special structure of the WH set, and in particular the duality between the shift operator and the modulation operator, the latter being absent in the context of general shift-invariant sets. In the third paper [RS3] we present applications of the results of [RS1] to wavelet (or affine) frames. The flavour of the results there is quite different; wavelet sets are not shift-invariant, and the main effort of [RS3] is to show that, nevertheless, the basic analysis of [RS1] does apply to that case as well.

Let $X \subset L_2(\mathbb{R}^d)$. We consider X as a possible "basis" set for $L_2(\mathbb{R}^d)$ or for some closed subspace of it. The various notions of a "basis set" are conveniently defined with the aid of the so-called *synthesis operator* or *reconstruction operator* $T := T_X$ defined by

$$T_X : \ell_0(X) \to L_2(\mathbb{R}^d) : c \mapsto \sum_{x \in X} c(x)x.$$

Here, $\ell_0(X)$ is the collection of all finitely supported sequences in $\ell_2(X)$. If T is bounded, it is extended by continuity to all of $\ell_2(X)$. We use the notation T for this extension, as well.

Definition 1.1. X is a *basis* whenever T is 1-1 on its domain. X is *fundamental* if $\operatorname{ran} T$ is dense in $L_2(\mathbb{R}^d)$. If T is bounded, X is a *Bessel set*. If T is bounded and $\operatorname{ran} T$ is closed, X is a *frame*. Finally, a frame that is also a basis is known as a *Riesz* (or *stable*) basis.

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