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EXPLICIT SIEGEL THEORY: AN ALGEBRAIC APPROACH

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To the memory of Martin Eichler

Let Q be a positive definite quadratic form on a $\mathbb{Z}$-lattice L of even rank $m \geqslant 6$; for convenience, assume $Q(L) \subseteq 2\mathbb{Z}$. To gain understanding of the representation numbers

$$r(L, 2n) = \#\{x \in L \colon Q(x) = 2n\},$$

we study the average representation numbers

$$r(\operatorname{gen} L, 2n) = \frac{1}{\operatorname{mass} L} \sum_{L' \in \operatorname{gen} L} \frac{1}{o(L')}\, r(L', 2n),$$

since $r(L, 2n)$ is asymptotic to $r(\operatorname{gen} L, 2n)$ as $n \to \infty$. Here L' runs over the distinct isometry classes within gen L, the genus of L; $o(L')$ denotes the order of the orthogonal group of L'; and mass $L = \sum_{L' \in \operatorname{gen} L}(1/o(L'))$.

In the 1930s Siegel used analytic methods to show that $r(\operatorname{gen} L, 2n)$ is a product of "p-adic densities' (see [5]; cf. [2]):

$$r(\operatorname{gen} L, 2n) = c \prod_{q} \frac{A_q(L, 2n)}{q^{m-1}},$$

where c is an easily computed constant, the product is over all $q = p^a$ with p prime and a sufficiently large, and $A_q(L, 2n)$ is the number of solutions to $Q(x) \equiv 2n \pmod{q}$, $x \in L/qL$. (Siegel actually shows that the average number of times a definite or indefinite quadratic form of arbitrary level and rank at least 4 represents another quadratic form is the product of p-adic densities.) One could use Hensel's lemma to compute the p-adic densities $((Aq(L, 2n))/(q^{m-1})$, but this gets extremely tedious when L is of arbitrary level.

We use algebraic considerations to obtain a new derivation of Siegel's formula, obtaining a more explicit formula for average representation numbers. We first consider lattices K whose associated theta series $\theta(K; \tau)$ have square-free, odd-level N, and quadratic character χ. Using local considerations, we design

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