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A HOMOTOPY-THEORETIC PROOF OF WILLIAMS'S METASTABLE POINCARÉ EMBEDDING THEOREM

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1. Introduction. Browder's concept of a *Poincaré embedding* (see [Br1], [Br3], [Br2], [Br4], [Wa], and [Ra]) is the homotopy analogue of a manifold embedding. Williams [Wi1] noted that a codimension-zero Poincaré embedding of a Poincaré pair (M, A) in S^m means only a map $f: A \to W$ such that $M \cup_f W$ is homotopy equivalent to S^m. He defined the *unstable normal invariant* as the composite $\rho: S^m \underset{\sim}{\xrightarrow{\text{orr.}}} M \cup_f W \to M/A$. Furthermore, two Poincaré embeddings $f: A \to W$ and $f': A \to W'$ of (M, A) are *concordant* if there exists $\alpha: W \to W'$ so that the maps $\alpha \circ f$, $f': A \to W'$ are homotopic. Williams [Wi1] proved the following geometrically, and outlined a program [Wi2] for a homotopy-theoretic proof.

THEOREM 1.1. *Let (M, A) be a finite, oriented Poincaré pair of dimension $m \geqslant 6$, and $\pi_1(A) \cong \pi_1(M)$. Suppose M is n-dimensional, and let $q = m - n - 1$. If $m \leqslant 3q$, then any degree-1 map $\rho: S^m \to M/A$ is induced by a Poincaré embedding $f: A \to W$. If $m < 3q$, the Poincaré embedding is unique up to concordance.*

We complete Williams's program here, via the elementary unstable homotopy theory of Barratt, Berstein-Hilton, Boardman-Steer, Ganea, James, and Toda.

Our homotopy proof here of Theorem 1.1 yields an example [Ri2] of the surgery machine in action: a manifold problem (in knot isotopy) is reduced via surgery theory to a homotopy theory problem, which is solved by homotopy-theoretic methods. We think this is the first time that the surgery machine has been used to solve a manifold problem when the underlying homotopy theory question was nontrivial.

Our knot applications are serendipitous; Williams aimed to understand *Poincaré surgery* (still not settled, although complicated geometric arguments have been proposed; see [Le] and [HV]), a problem designed to test the efficacy of the surgery machine. Quinn [Qu] claimed that producing Poincaré embeddings of a sphere in a Poincaré complex in or below the middle dimension should be a metastable desuspension problem. Williams [Wi2] noticed that the "opposite" problem above actually was metastable. Via the fiberwise homotopy theory ideas of Weinberger and Klein, our metastable proofs here appear to generalize to Poincaré surgery. Ideas of Williams seem to extend the proofs to give Poincaré handlebody decompositions.

Williams does not have to assume that (W, A) is a Poincaré pair, because (see

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