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NORMALITY WITH RESPECT TO POWERS OF A BASE

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1. Introduction. Classically, a real number x is *normal in base* r, where $r \geqslant 2$ is an integer, if $\{r^n x\}$ is uniformly distributed modulo 1. This concept may also be expressed in terms of digit expansions in base r and, indeed, the original definition of normality by Borel was in these terms. The concern of this paper is normality with respect to two different powers of a fixed base. For integer bases the general theory is complete. Maxfield [15] has shown that normality in base r^p and in base r^q are equivalent when p and q are any positive integers. He also proved that if x is normal in base r, then so is cx for any rational number c. Schmidt [24], [25] proved the converse of the first of Maxfield's results, that is, that normality to two different integer bases is the same if and only if they are powers of each other.

The situation for noninteger bases is less well understood. Even the definition of normality has several different forms. It is no longer the case that the digit definition of normality coincides with the uniform distribution definition. A more complete discussion of the various concepts of normality in this context can be found in Mendès France [16], [17] and Bertrand [1], [2], [3]. Following Mendès France, we define

$$B(\theta) = \{x \in \mathbf{R}\colon \{\theta^n x\} \text{ is uniformly distributed modulo } 1\}.$$

Here we shall restrict our attention to the uniform distribution definition.

In [16] Mendès France discusses the possibility of extending the results of Maxfield to the noninteger case. In fact, he asks the following questions:

(1) Is $B(\theta) = (1/q)B(\theta)$ for $q > 1$ an integer?

(2) Does $B(\theta) = B(\phi)$ if and only if $\log\theta/\log\phi$ is rational?

For (1), it is enough to show that $(1/q)B(\theta) \subset B(\theta)$, since the reverse inclusion follows immediately from the Weyl criterion. Mendès France also makes a conjecture that (2) is true if and only if θ is a Pisot-Vijayaraghavan (PV) number. The first question is also asked by Rauzy in [23]. These questions and related ones were taken up by Bertrand in her thesis [1]. She proved, among other results, the following theorem.

THEOREM (Bertrand). *If*

$$\theta^k \pm \theta^{-k} \in \mathbf{Z},$$

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