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STRONG LAWS FOR WEIGHTED SUMS OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES

I. ASSANI

A. Introduction. Let (a_n), $n \geqslant 1$ be a sequence of positive real numbers, $A_n = \sum_{k=1}^{n} a_k$ for all $n \geqslant 1$, and let $\{Z_k; k \geqslant 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables such that $E(|Z_1|) < \infty$. Under these assumptions, B. Jamison, S. Orey, and W. Pruitt ([JOP]; see also W. Stout [So]) obtained a necessary and sufficient condition for the a.e. convergence of the sequence $T_n = (\sum_{n=1}^{N} a_k Z_k(y))/A_n$. More precisely, if $N_n = \#\{a_k/A_k \geqslant 1/n\}$, the condition $\sup_n (N_n/n) < \infty$ is necessary and sufficient for the a.e. convergence of the sequence T_n.

One of our results in this paper is the following multidimensional form of Kolmogorov's strong law of large numbers.

THEOREM 1. *Let H be a positive integer. Given an i.i.d. sequence $(X_{1n})_n$ on the probability measure space $(\Omega_1, \mathscr{F}_1, \mu_1)$ satisfying $E(|X_{1,1}|) < \infty$, it is possible to find a set of full measure $\tilde{\Omega}_1$ such that if $x_1 \in \tilde{\Omega}_1$, the following holds.*

(1) *For all probability measure spaces $(\Omega_2, \mathscr{F}_2, \mu_2)$ and all i.i.d. sequences $(X_{2,n})_n$ such that $E(|X_{2,1}|) < \infty$, it is possible to find a set of full measure $\tilde{\Omega}_2$ such that if $x_2 \in \tilde{\Omega}_2$, the following holds.*

(2)

⋮

$(H-1)$ *For all probability measure spaces $(\Omega_H, \mathscr{F}_H, \mu_H)$ and all i.i.d. sequences $(X_{H,n})_n$ such that $E(|X_{H,1}|) < \infty$, we find a set of full measure $\tilde{\Omega}_H$ for which, if $x_H \in \tilde{\Omega}_H$, we have*

$$\frac{1}{N}\sum_{n=1}^{N} X_{1,n}(x_1) X_{2,n}(x_2) \cdots X_{H,n}(x_H) \underset{N}{\rightarrow} \prod_{i=1}^{H} E(X_{i,1}). \quad \square$$

The case $H = 1$ corresponds to the strong law for large numbers. The difficulty here is that at each stage the null set does not depend on the incoming i.i.d. sequences and the associated probability spaces. We first obtained the case $H = 2$, $\mathbb{E}(|X_{1,1}|^3) < \infty$, and then prior to our proof of Theorem 1, a proof in the case $H = 2$, $E(|X_{1,1}|^2) < \infty$, was shown to us by Gordon Simons, using the notion of complete convergence. We give his argument in the appendix.

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