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NONLINEAR WAVE EQUATIONS: CONSTRAINTS ON PERIODS AND EXPONENTIAL BOUNDS FOR PERIODIC SOLUTIONS

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1. Introduction. In this article, we study periodic solutions of the nonlinear wave equation (NLW)

$$\partial_t^2 \varphi - \Delta\varphi + f(\varphi) = 0, \tag{1.1}$$

where $\varphi\colon \mathbb{R}_x^N \times \mathbb{R}_t \to \mathbb{R}, f\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$, and $\partial_t^2 = \partial^2/\partial t^2$, $\Delta = \sum_{i=1}^N \partial^2/\partial x_i{}^2$. By a periodic solution, we understand solutions that are periodic in time t, and L^2 in x. This notion extends, on the one hand, the concepts of bound states of the Schrödinger equation and standing waves of linear wave equations and, on the other hand, the concept of periodic solutions of dynamical systems. (Equation (1.1) can be viewed as an infinite-dimensional Hamiltonian system.) Both concepts are among the simplest and most basic in the fields mentioned.

To state our results, we introduce some notation. Let S_ω^1 denote the circle of radius ω^{-1}. The class of solutions we consider is the following set:

$$\mathscr{D}_\omega \equiv \Big\{\varphi \in H^1(\mathbb{R}^N \times S_\omega^1);\ \text{if } \psi \text{ is any of } \varphi,\ \partial_t\varphi, \text{ or } x\cdot\nabla\varphi, \text{ then } \|\psi\|_{L^\infty(\mathbb{R}^N \times S_\omega^1)} < \infty$$

$$\text{and} \lim_{|x|\to\infty} |\psi(x,t)| = 0 \text{ uniformly in } t \Big\}. \tag{1.2}$$

(This class of solutions can probably be enlarged.) Here $H^1(\Omega)$ stands for the Sobolev space of order 1 for functions on Ω.

Our main result is a characterization of two fundamental properties of periodic solutions: their frequencies and their spatial localization. Consequently, we find that the spatial and temporal properties of periodic solutions are related. More precisely, we prove the following theorems.

THEOREM 1.1. *Suppose $f \in C^3(\mathbb{R}, \mathbb{R})$. Let φ be a nontrivial $2\pi/\omega$-periodic solution of NLW on $\mathscr{D}_\omega$. Then $\omega^2 \leqslant f'(0)$.*

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