Vol. 88, No. 1 DUKE MATHEMATICAL JOURNAL © 1997

ELLIPTIC FACTORS OF SELBERG ZETA FUNCTIONS

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§1. Introduction. Let $X = G/K$ be a rank-one Riemannian symmetric space of noncompact type, with G a connected semisimple real Lie group of real rank one and K its maximal compact subgroup. Let Γ be a uniform lattice of G, that is, a discrete cocompact subgroup of G. We do *not* exclude the possibility that Γ contains elements of finite order greater than one. Thus the natural action of Γ on X may not be fixed-point free. For every $\gamma \in \Gamma$ or its Γ-conjugacy class $[\gamma]$, we define its *norm* $N(\gamma)$ by setting $N(\gamma) = \sup_\lambda |\lambda|^{1/2}$ with λ ranging over the eigenvalues of $\mathrm{Ad}(\gamma)$ on $\mathfrak{g}_{\mathbb{C}}$, the complexified Lie algebra of G.

For every pair (X, Γ) as stated above, we can attach the Selberg-type zeta function, which is at first given on some half-plane by a convergent Euler product

$$Z_\Gamma(s) = \exp\left(\sum_{\gamma \in \mathscr{H}(\Gamma)} \frac{a(\gamma)}{\det(1 - \mathrm{Ad}(\gamma^{-1})|\mathfrak{n}[\gamma])} N(\gamma)^{-s} \right),$$

where $\mathscr{H}(\Gamma)$ denotes the set of conjugacy classes $[\gamma]$ of Γ such that $N(\gamma) > 1$, $a(\gamma)$ is a certain volume factor prescribed to each conjugacy class $[\gamma] \in \mathscr{H}(\Gamma)$, and $\mathfrak{n}[\gamma]$ is the subspace of $\mathfrak{g}$ consisting of all $Y \in \mathfrak{g}$ such that $\lim_{n \to +\infty} \mathrm{Ad}(\gamma)^{-n} Y = 0$. This type of zeta function, which was originally studied by Selberg for X, the upper half-plane, was investigated by Gangolli [G2] for a general pair (X, Γ) under the assumption that Γ is torsion-free. He proved that $Z_\Gamma(s)$ is meromorphically continued to the whole s-plane and satisfies a functional equation. Furthermore, the divisor of the meromorphic function $Z_\Gamma(s)$ is described in terms of the spectrum of the Laplace-Beltrami operator Δ_Γ acting on the L^2-space of the locally symmetric manifold $\Gamma\backslash X$. The last-mentioned relationship of the zeta function $Z_\Gamma(s)$ and the differential operator Δ_Γ is grasped more directly and explicitly by considering the *regularized determinant* of $\Delta_\Gamma + s(s - 2\rho_0)$ for an appropriate constant ρ_0. Indeed, for X, the upper half-plane, and Γ, the torsion-free uniform lattice of $PSL_2(\mathbb{R})$, Sarnak [Sa] and Voros [Vo] proved that the regularized determinant of $\Delta_\Gamma + s(s-1)$ essentially equals $Z_\Gamma(s)$ multiplied by some meromorphic function $G_\Gamma(s)$ that is explicitly calculated by means of double gamma functions. The meromorphic function $G_\Gamma(s)$ is regarded as a kind of gamma factor of $Z_\Gamma(s)$ in analogy with the Riemann zeta function.

Our problem is to obtain the analogous result for a general pair (X, Γ). In

Received 28 May 1996.