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ENUMERATIVE GEOMETRY FOR THE REAL GRASSMANNIAN OF LINES IN PROJECTIVE SPACE

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1. Introduction. Describing the common zeroes of a set of polynomials is more problematic over nonalgebraically closed fields. For systems of polynomials with few monomials on a complex torus ("fewnomials"), Khovanskii [8] showed that the number of real zeroes is at most a small fraction of the number of complex zeroes. Fulton [5, §7.2] asked how many solutions to a problem of enumerative geometry can be real; for example, how many of the 3,264 conics tangent to 5 general real conics can be real? He later showed that all, in fact, can be real. This was rediscovered by Ronga, Tognoli, and Vust [13]. Robert Speiser suggested the classical Schubert calculus of enumerative geometry would be a good testing ground for this question. For problems of enumerating lines in $\mathbf{P}^n$ incident to real linear subspaces in general position, we show that all solutions can be real.

Let $\mathbf{G}_1\mathbf{P}^n$ be the Grassmannian of lines in $\mathbf{P}^n$. A flag and a partition $\lambda = (\alpha, \beta)$ determine a Schubert subvariety of type λ, which has codimension $|\lambda| = \alpha + \beta$. The automorphism group PGL_{n+1} of $\mathbf{P}^n$ acts transitively on $\mathbf{G}_1\mathbf{P}^n$ and on the set of Schubert varieties of a fixed type. Over fields k of characteristic zero, Kleiman's transversality theorem [9] shows that a general collection of Schubert subvarieties of $\mathbf{G}_1\mathbf{P}^n$ will intersect generically transversally. In §5, we extend this to fields of positive characteristic. A consequence is that for Schubert-type enumerative problems in $\mathbf{G}_1\mathbf{P}^n$, the basic principle of the Schubert calculus remains valid in positive characteristic: each component of a general intersection of Schubert varieties appears with multiplicity one. For partitions $\lambda^1, \ldots, \lambda^m$, let $\mathscr{G}(\lambda^1, \ldots, \lambda^m)$ be the (nonempty) set of points of the Chow variety of $\mathbf{G}_1\mathbf{P}^n$ representing cycles arising as generically transverse intersections of Schubert varieties of types $\lambda^1, \ldots, \lambda^m$. Any generically transverse intersection of Schubert varieties is rationally equivalent to a sum of Schubert varieties; the Schubert calculus gives algorithms for determining how many of each type. In §4, we prove the following theorem.

THEOREM A. *Let $\lambda^1, \ldots, \lambda^m$ be partitions. Then there is a cycle Φ (depending upon $\lambda^1, \ldots, \lambda^m$) whose components are explicitly described Schubert varieties, such that Φ is in the Zariski closure of $\mathscr{G}(\lambda^1, \ldots, \lambda^m)$. Moreover, for each cycle X*

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