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SOME REMARKS ON MASSEY PRODUCTS, TIED COHOMOLOGY CLASSES, AND THE LUSTERNIK-SHNIRELMAN CATEGORY

CLAUDE VITERBO

1. Introduction. Let X be a Hilbert manifold, $F(X)$ the minimal number of critical points for a function on X, and $c\ell(X)$ the cup length of X, that is, the maximal integer k, such that there are classes $\beta_1, \dots, \beta_{k-1}$ in $H^*(X) - H^0(X)$ with nonzero cup product. Then it is classical that $c\ell(X) \leqslant F(X)$.

In Section 2, we introduce the notion of tied cohomology classes and use it to improve the cup length into a "tie length" $T\ell(X)$, a lower bound for $F(X)$. We then estimate $T\ell(X)$ using in particular Massey products.

In Section 4 we deal with the "stable tie length" $t\ell(X)$ and again give some estimates. Applications to symplectic geometry and examples where $t\ell(X) > c\ell(X)$ are given in the last section.

Let f be a C^1 function on a Hilbert manifold X, satisfying the Palais-Smale condition (PS): Any sequence (x_n) such that $f'(x_n) \to 0$ and $f(x_n)$ is bounded has a converging subsequence.

Set for $a \in \mathbb{R}$, $X^a = \{x \in X | f(x) \leqslant a\}$. Note that for $-\infty < a \leqslant b < +\infty$, (PS) implies that the set of critical points in the closure of $X^b - X^a$ is compact, and the set of critical values is closed.

Let $F(X)$ be the minimal number of critical points for a function bounded from below and satisfying (PS) on X. (If X has a boundary ∂X, we assume $\lim_{x \to \partial X} f(x) = +\infty$.) More generally, for A submanifold (with boundary) of X, $F(X, A)$ is defined as the minimal number of critical points in $X - A$ for a nonnegative function X, satisfying (PS) and vanishing on A. Changing f to $1/f$, we see that it is also the number of critical points for a function f on $X - A$ such that $\lim_{x \to A} f(x) = +\infty$.

A lower bound for $F(X)$ was given by Lusternik and Shnirelman [LuS], denoted by $\operatorname{cat}(X)$; it is the minimal number of open subsets, contractible in X, needed to cover X. Note that the definition of $\operatorname{cat}(X)$; is valid on any topological space, not just a manifold.

A well-known consequence of Lusternik and Shnirelman's work is that, denoting by $c\ell(X)$ (= cup length of X) the number

$$c\ell(X) = \max\{k | \exists \beta_1, \dots, \beta_{k-1} \in H^*(X) - H^0(X), \beta_1 \cdot \beta_2 \cdots \beta_{k-1} \neq 0\},$$

then $\operatorname{cat}(X) \geqslant c\ell(X)$. Note that $c\ell(X)$ depends on the choice of the coefficient ring.

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