Vol. 86, No. 2 DUKE MATHEMATICAL JOURNAL © 1997

CIRCLE PATTERNS WITH THE COMBINATORICS OF THE SQUARE GRID

ODED SCHRAMM

1. Introduction. Some aspects of the relatively new theory of circle patterns (or packings) and their relations to analytic functions can be considered well understood, while other aspects are still very mysterious and enigmatic. The roots of the topic are in the circle packing theorem [18], which is a uniformization result. The uniformization theory of circle patterns is now well developed [29], [1], [16]. There also has been steady progress in understanding the convergence of circle packings to conformal maps [22], [12], [7], [17]. Branched packings were considered, and analogues of polynomials and finite Blaschke products have been studied [3], [9], [10]. Also, a satisfying crop of applications and by-products has emerged [21], [4], [20], [24], [15], [13], [2].

However, there is also a darker side: very little is known about circle pattern analogues of entire functions. It seems that Peter Doyle was the first to look into this area; he constructed entire immersed hexagonal circle patterns analogous to the exponential map. Doyle conjectured that these immersed packings, which came to be known as Doyle spirals, are the only entire immersed hexagonal circle packings. To date, the only other contribution to this topic seems to be Callahan and Rodin's [5]. They showed that entire immersed hexagonal circle packings form regularly exhaustible surfaces, and therefore satisfy Ahlfors's value distribution theory. In particular, Picard's theorem is valid in this setting.

We have found that a framework of circle patterns based on the square grid, which we call *SG* patterns, is more tractable than the traditional hexagonal patterns. It turns out that Doyle's conjecture is false for *SG* patterns: there is an *SG* pattern analogous to the entire function $\operatorname{erf}(\sqrt{i}z)$, where $\operatorname{erf}(z) = \int^z e^{-w^2}\,dw$ is the error function. We also show that the collection of entire *SG* patterns in the sphere is infinite-dimensional and exhibit some explicit finite-dimensional families. "Explicit" means that there is a closed form expression for the radius of any given circle in the pattern. Quite possibly, more explicit finite-dimensional families could be found. This would be very interesting.

In joint work, yet unpublished, with Rick Kenyon, we have discovered explicit branched *SG* patterns analogous to polynomials and to the function $\log z$.

Much of the theory of the hexagonal packings can be carried over to *SG*

Received 13 March 1996.

Some of the work exposed here was done while the author visited the University of California–San Diego, and the author wishes to express thanks to the members of the UCSD Mathematics Department for their hospitality. This research was partially supported by National Science Foundation grant DMS94-03548.