Vol. 86, No. 2 DUKE MATHEMATICAL JOURNAL

POINTWISE ERGODIC THEOREMS FOR RADIAL AVERAGES ON SIMPLE LIE GROUPS II

AMOS NEVO

§1. Statement of results, the method of proof, and some remarks

1.1. Definitions and statement of results. The present paper is a continuation of [N1], and we begin by briefly recalling the setup and the notation:

- $G = G_n = SO^0(n, 1)$ is the group of orientation-preserving isometries of n-dimensional real hyperbolic space H^n, $n \geqslant 2$.
- K = a fixed maximal compact subgroup. m_K = Haar probability measure.
- $A = \{a_t | t \in \mathbb{R}\}$ = a one-parameter group of hyperbolic translations such that $G = KA_+K$ is a Cartan decomposition.
- σ_t = the bi-K-invariant probability measure on G given by $\sigma_t = m_K * \delta_{a_t} * m_K$, where $*$ denotes convolution. Note that $\sigma_0 = m_K$.
- $\mu_t = 1/t \int_0^t \sigma_s \, ds$, the uniform average of $\sigma_s, 0 \leqslant s \leqslant t$. We define $\mu_0 = m_K$.
- $M(G, K)$ = the commutative convolution algebra (of bi-K-invariant complex bounded Borel measures on G) generated by $\sigma_t, t \geqslant 0$.
- $(X, \mathscr{B}, \lambda)$ = a standard Borel space with a Borel measurable G-action which preserves the probability measure λ.
- $\pi(\nu)f(x) = \int_G f(g^{-1}x) \, d\nu(g)$ = the Markov operator on $L^p(X)$ corresponding to a probability measure ν on G.
- $M_\mu f(x) = \sup_{t \geqslant 0} |\pi(\mu_t)f(x)|$, and $M_\sigma f(x) = \sup_{t \geqslant 0} |\pi(\sigma_t)f(x)|$, maximal functions associated with the action of σ_t and μ_t in $L^p(X)$, $1 \leqslant p \leqslant \infty$.

Finally, recall also the following definition.

Definition. Let ν_t, $t \geqslant 0$, be a one-parameter family of probability measures on G. Assume that $t \mapsto \nu_t \in M(G)$ is continuous in the w^*-topology of $M(G)$ as the dual of $C_0(G)$. Let $(X, \mathscr{B}, \lambda)$ denote a G-space as above.

(1) ν_t is called a *pointwise ergodic family in* L^p if, for any $f \in L^p(X)$,

$$\lim_{t \to \infty} \pi(\nu_t)f(x) = E_1(f)(x),$$

where the convergence is pointwise almost everywhere and in the L^p-norm, and E_1 is the conditional expectation of f with respect to the σ-algebra of G-invariant sets.

(2) ν_t is said to satisfy the *local ergodic theorem in* L^p if, for any $f \in L^p(X)$, $\lim_{t \to 0} \pi(\nu_t)f(x) = \pi(\nu_0)f(x)$, where the convergence is for almost every x, and in the L^p-norm.

Received 18 May 1995. Revision received 26 March 1996.