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QUADRATIC FORMS FOR A 2-D SEMILINEAR SCHRÖDINGER EQUATION

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1. Introduction. Part of this paper is a generalization in 2-D of the work of Kenig, Ponce, and Vega on local well-posedness in Sobolev spaces with negative indices for a semilinear Schrödinger equation [9].

The problem we consider is a special case of the semilinear IVP (initial value problem)

$$\begin{cases} i\partial_t u + \Delta u + N(u, \bar{u}) = 0 \\ u(x, 0) = \Phi(x) \qquad x \in \mathbb{R}^2,\ t \in \mathbb{R}, \end{cases} \tag{1}$$

where $N(u, \bar{u}) = \sum_{|\alpha|=k} a_\alpha u^{\alpha_1} \bar{u}^{\alpha_2}$ and $\Phi \in H^s$. The IVP (1) has been extensively studied; see, for example, [4], [5], [6], [7], and [12]. In particular, T. Cazenave and F. Weissler proved that (1) is well-posed in L^2 whenever $k \leqslant 3$. To prove this result, the authors use Strichartz-type estimates [11] and only the homogeneity of the potential; no other structure of $N(u, \bar{u})$ is required. On the other hand, a scaling argument, like the one presented in [9], suggests that for nonlinearities $N(\cdot)$ of lower degree, one may expect local well-posedness results also in H^s, $-1 < s \leqslant 0$. We show in this paper that this conjecture can be proved in some cases. More precisely, we show that the IVP (1) with $N(u, \bar{u}) = \bar{u}^2$ is locally well-posed in H^s, $s > -1/2$ (see Theorem 2.4). Unfortunately our method depends not just on the degree of the nonlinearity, but also on the structure of it. In fact, the key estimate we use in the proof (see Definition 1.1, Theorem 2.1 and also [9]),

$$\|\bar{u}^2\|_{X^{s,1-b}} \leqslant C\|u\|^2_{X^{s,b}}, \qquad \text{in } \mathbb{R}^2 \times \mathbb{R} \tag{2}$$

for $s \in (-1/2, 0]$ and some $b > 1/2$, is not true if we replace the left-hand side with $\|\bar{u}u\|_{X^{s,1-b}}$, and no results have been proved yet for $\|u^2\|_{X^{s,1-b}}$ with $s < 0$. This phenomenon was already observed by Kenig , Ponce, and Vega for the IVP (1) on the line and on the circle. Using an inequality like (2) on $\mathbb{R} \times \mathbb{R}$ and on $\mathbb{T} \times \mathbb{R}$, they proved that the IVP (1) with nonlinearity u^2 or $\bar{u}^2$ is locally well-posed in H^s, $s > -3/4$, if we are on the line, and in H^s, $s > -1/2$, if we are on the circle. On the other hand, if we consider the nonlinearity $\bar{u}u$, then on the line the IVP (1) is locally well-posed in H^s, $s > -1/4$, and on the circle no results are known for $s < 0$.

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