Vol. 86, No. 1 DUKE MATHEMATICAL JOURNAL © 1997

A HILBERT SPACE OF DIRICHLET SERIES AND SYSTEMS OF DILATED FUNCTIONS IN $L^2(0,1)$

HÅKAN HEDENMALM, PETER LINDQVIST, AND KRISTIAN SEIP

1. Introduction. The purpose of this paper is twofold. First, we study systems of functions of the form $\varphi(x), \varphi(2x), \varphi(3x), \ldots$, and second, we consider the Hardy space H^2 of the infinite-dimensional polydisk. Building on ideas of Arne Beurling and Harald Bohr, we find that the two topics are intimately connected, the common feature being the use of Dirichlet series.

Let $\varphi \in L^2(0,1)$ be given and consider φ as defined on the whole real axis by extending it to an odd periodic function of period 2. The Riesz-Fischer theorem of Fourier analysis states that for $\varphi(x) = \sqrt{2}\sin(\pi x)$ the sequence $\varphi(nx)$, $n = 1, 2, 3, \ldots$, is an orthonormal basis in the Hilbert space $L^2(0,1)$. The question raised in this paper is which functions can take the place of the sine in this theorem. It is clear that the statement must be weakened, because the only orthogonal bases are obtained from $\varphi(x) = C\sin(\pi x)$. If we instead ask for a classification of those φ for which the system $\{\varphi(nx)\}_n$ is a Riesz basis (a basis orthonormal with respect to an equivalent norm) or of those φ for which the same system is a complete sequence in $L^2(0,1)$, we are led to profound problems.

The latter of the two problems—the completeness problem—was stated by Beurling in his seminar on harmonic analysis in Uppsala in 1945. A brief note from this seminar is found in [1]. Beurling's note indicates that a natural way to approach these problems is to associate to the given function

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \sqrt{2}\sin(n\pi x)$$

the Dirichlet series

$$S\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \tag{1-1}$$

and to try to express the Riesz basis and completeness properties in terms of analytic properties of $S\varphi(s)$. This approach has proved fruitful. We have solved

Received 11 December 1995.

Hedenmalm's work was supported in part by the Swedish Natural Science Research Council. Lindqvist and Seip were supported in part by the Norwegian Research Council. While at the Mathematical Sciences Research Institute, Berkeley, California, Hedenmalm and Seip were also supported in part by National Science Foundation grant DMS-9022140.