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SYMPLECTIC COUPLES ON 4-MANIFOLDS

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1. Introduction. Let M be a (smooth, connected, oriented) 4-manifold. Recall that a symplectic form on M is a closed, nondegenerate differential 2-form ω, where nondegeneracy means that ω^2 is a volume form on M. In the present paper, we consider the following structure.

Definition 1.1. (i) A pair of symplectic forms (ω_1, ω_2) on M is called a *symplectic couple* if $\omega_1 \wedge \omega_2 \equiv 0$ and ω_1^2, ω_2^2 are volume forms defining the positive orientation.

(ii) We say that three symplectic forms $(\omega_1, \omega_2, \omega_3)$ constitute a *symplectic triple* if $\omega_i \wedge \omega_j \equiv 0$ for $i \neq j$ and the ω_i^2 are positive volume forms.

(iii) A symplectic couple (triple) is called *conformal* if $\omega_1^2 = \omega_2^2\ (= \omega_3^2)$.

Our motivation to study symplectic couples and triples is twofold. First, observe that if (ω_1, ω_2) is a symplectic couple, then any nontrivial linear combination $\lambda_1\omega_1 + \lambda_2\omega_2$, $(\lambda_1, \lambda_2) \in \mathbb{R}^2 - (0,0)$, is again a symplectic form. Furthermore, if (ω_1, ω_2) is conformal, then $\lambda_1\omega_1 + \lambda_2\omega_2$ defines the same volume form as ω_1 and ω_2 for any (λ_1, λ_2) on the unit circle S^1 in $\mathbb{R}^2$.

Pairs of contact forms (α_1, α_2) with the analogous properties are investigated in [5], and the following is proved there.

THEOREM 1.2. *Let N be a closed 3-manifold. Then N admits a pair of contact forms (α_1, α_2) such that $\lambda_1\alpha_1 + \lambda_2\alpha_2$ is a contact form defining the same volume form for any $(\lambda_1, \lambda_2) \in S^1$ (equivalently, $\alpha_1 \wedge d\alpha_1 = \alpha_2 \wedge d\alpha_2$ and $\alpha_1 \wedge d\alpha_2 = -\alpha_2 \wedge d\alpha_1$) if and only if N is diffeomorphic to a quotient of the Lie group $\mathscr{G}$ under a discrete subgroup acting by left multiplication, where $\mathscr{G}$ is one of the following:*

(a) *$S^3 = SU(2)$, the universal cover of $SO(3)$,*

(b) *$\widetilde{SL}_2$, the universal cover of $PSL_2\mathbb{R}$,*

(c) *$\tilde{E}_2$, the universal cover of the Euclidean group (of orientation-preserving isometries of $\mathbb{R}^2$).*

Second, our definition of symplectic couple is motivated by the recent work of Lucas Hsu [9], where the condition $\omega_1 \wedge \omega_2 \equiv 0$ is equivalent to a certain pair of first-order partial differential equations, whose solution surfaces are Lagrangian with respect to both ω_1 and ω_2, to form a system of Euler-Lagrange type.

This second motivation is our main reason for studying symplectic couples

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