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ON EXTENDABILITY OF ISOMETRIC IMMERSIONS OF SPHERES

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1. Introduction. A central problem from classical differential geometry consists of classifying isometric immersions from Riemannian manifolds of constant curvature into Euclidean space. For example, it was proven by É. Cartan that there are no n-dimensional submanifolds of constant negative curvature in E^{2n-2} and the submanifolds of constant negative curvature in E^{2n-1} depend upon $n(n-1)$ functions of a single variable (see [5, §7]). In the case of constant zero curvature, Hartmann and O'Neill showed that isometric immersions of E^n into E^{2n-k} must be k-cylindrical when $k \leqslant n$ (see [3, §5.2]).

In the positive curvature case, it is well known that the standard round n-sphere S^n of constant curvature one is rigid in E^{n+1} when $n \geqslant 2$. However, it is not rigid in E^{n+2}, since one can construct an infinite-dimensional family of compositions of isometric immersions

$$S^n \to E^{n+1} \to E^{n+2}.$$

The following theorem shows that this is essentially the only way in which rigidity fails when n is at least three.

THEOREM. *Suppose that S^n is the n-dimensional sphere of constant curvature one, regarded in the usual fashion as the boundary of the flat unit disk D^{n+1}. If $n \geqslant 3$ and $F: S^n \to E^{n+2}$ is a C^∞ isometric immersion, there exists a unique C^∞ isometric immersion $\tilde{F}: D^{n+1} \to E^{n+2}$ such that $\tilde{F}|S^n = F$.*

In the case where $n \geqslant 4$, our argument relies on an algebraic lemma of O'Neill [11]. The case $n = 3$ is deeper, since there is a richer supply of local solutions to the isometric immersion problem; in this case the algebraic lemma of O'Neill does not suffice, and we need to appeal to Theorem 3 from [8].

When $n \geqslant 4$, local versions of the above theorem were obtained by Erbacher [4] and Henke [6]. The main technical difficulty in proving a global theorem consists of establishing smoothness of $\tilde{F}$ at the umbilic points of F. (By definition, an *umbilic point* of an isometric immersion $S^n \to E^{n+2}$ is a point at which the second fundamental form looks exactly like that of a standard isometric immersion into a hyperplane.) For this purpose, we use a surprisingly subtle theorem of Joris [7], which asserts that if the square and cube of a real-valued function are smooth, then the function itself is smooth.

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