Vol. 85, No. 3 DUKE MATHEMATICAL JOURNAL © December 1996

ON DEGENERATE SECANT AND TANGENTIAL VARIETIES AND LOCAL DIFFERENTIAL GEOMETRY

J. M. LANDSBERG

§1. Introduction and conventions. One way to study geometric properties of a variety $X^n \subset \mathbb{CP}^{n+a}$ is by studying coarse geometric properties of auxiliary varieties one constructs from X. The auxiliary varieties we will study in this paper are the secant variety $\sigma(X)$ and the tangential variety $\tau(X)$, and the coarse properties of $\sigma(X)$ and $\tau(X)$ we will study are their dimensions. For information on how this study fits into larger questions, see [LV]. It turns out that smooth varieties of small codimension with degenerate $\tau(X)$ (degenerate meaning that $\tau(X)$ is not the entire ambient space) carry a remarkable amount of infinitesimal geometric structure. Before going into details, we will need a few definitions.

Given a variety $X^n \subset \mathbb{P}^{n+a}$, the *secant variety* $\sigma(X)$ of X is defined to be the union of all points on all secant and tangent lines (i.e., $\mathbb{P}^1$'s) of X. More precisely, given $p, q \in \mathbb{P}^{n+a}$, let $\mathbb{P}^1_{pq} \subset \mathbb{P}^{n+a}$ denote the projective line containing p and q. Then

$$\sigma(X) := \overline{\{x \in \mathbb{P}^{n+a} | x \in \mathbb{P}^1_{pq} \text{ for some } p, q \in X\}}.$$

Secant varieties have been studied extensively. Two important results on them are the following.

THEOREM 1.1 (Zak's theorem on linear normality [FL], [Z1]). *Let $X^n \subset \mathbb{CP}^{n+a}$ be a smooth variety not contained in a hyperplane with $\sigma(X) \neq \mathbb{CP}^{n+a}$. Then $a \geqslant (n/2) + 2$.*

THEOREM 1.2 (Zak's theorem on Severi varieties [LV], [Z1]). *Let $X^n \subset \mathbb{CP}^{n+a}$ be a smooth variety not contained in a hyperplane with $\sigma(X) \neq \mathbb{CP}^{n+a}$. If $a = (n/2) + 2$, then X is one of the following:*

(i) *Veronese $\mathbb{P}^2 \subset \mathbb{P}^5$;*
(ii) *Segre $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;*
(iii) *Plücker embedded Grassmannian $G(\mathbb{C}^2, \mathbb{C}^6) \subset \mathbb{P}^{14}$;*
(iv) *$E_6/P \subset \mathbb{P}^{26}$.*

These four varieties, now called *Severi varieties*, also have other special properties. For example, they classify the quadro-quadro Cremona transforms (see [ESB]). We give a new proof of Theorem 1.2 via local differential geometry.

Received 13 December 1994. Revision received 7 September 1995.
This work was done while the author was partially supported by a National Science Foundation Postdoctoral Fellowship and later by National Science Foundation grant DMS-9303704.