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HASSE PRINCIPLE FOR WITT GROUPS OF FUNCTION FIELDS WITH SPECIAL REFERENCE TO ELLIPTIC CURVES

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APPENDIX BY J.-L. COLLIOT-THÉLÈNE

With gratitude to Sridharan on his 60th birthday

Let k be a global field, char $k \neq 2$. Let X be a smooth, projective geometrically integral curve over k such that $X(k) \neq \varnothing$. Let J denote the Jacobian of X and ${}_2Ш(J)$ be the 2-torsion subgroup of the Tate-Shafarevich group of the Jacobian of X. We denote the set of places of k by $P(k)$ and the completion at $v \in P(k)$ by k_v. Let $K = k(X)$ be the function field of X and $k_v(X)$ be the function field of $X_v = X \times_k k_v$. It is shown in [S] that the kernel of the natural map

$$h\colon W(k(X)) \to \prod_{v \in P(k)} W(k_v(X))$$

injects into ${}_2Ш(J)$, where for any field F, $W(F)$ denotes the Witt group of quadratic forms over F. In this paper, we prove that this injection is in fact an isomorphism (§1). Thus, the obstruction to the Hasse principle for Witt groups of function fields of curves (with a k-point) is detected completely by the 2-torsion of the Tate-Shafarevich group of the Jacobian. We show by an example that the condition $X(k) \neq \varnothing$ is in fact essential (§5).

We analyse this correspondence between the obstruction to the Hasse principle and elements of ${}_2Ш(E)$ more explicitly in the case of an elliptic curve E (§2, §3). We show that for an elliptic curve, every element in the kernel of h is given by the norm form of a quaternion algebra, unramified on E. In fact, we give an explicit description (§3) of such forms in the kernel, for elliptic curves over $\mathbf{Q}$ of the form $y^2 = x^3 - Dx$, given an element of ${}_2Ш(E)$. These elements correspond to smooth conic fibrations on E which are locally (i.e., over the completions) generically trivial.

Classically, the first example of a genus one curve over $\mathbf{Q}$, which was a counterexample to the Hasse principle for varieties, was constructed by Reichardt and Lind [R]. Using the description of such a space as an intersection of two quadrics (in a specific form) in $\mathbf{P}^3$, Colliot-Thélène gives a procedure (§3) to construct a smooth conic fibration of the above kind on an elliptic curve. Our theorem asserts that this is indeed the case *in general*: every principal homogeneous

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