Vol. 85, No. 2 DUKE MATHEMATICAL JOURNAL © November 1996

A NEW ISOPERIMETRIC COMPARISON THEOREM FOR SURFACES OF VARIABLE CURVATURE

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§0. Introduction. In this paper, we consider isoperimetric profiles of Riemannian surfaces with variable curvature. The isoperimetric profile of a Riemannian manifold M^n is the function $I_{M^n}\colon [0, \mathrm{vol}(M^n)) \to R_+$ defined by

$$I_{M^n}(v) = \inf\{\mathrm{vol}_{n-1}(\partial\Omega) \big| \Omega \subset M^n \text{ a compact domain with smooth boundary } \partial\Omega, \mathrm{vol}(\Omega) = v\}.$$

In general, the isoperimetric profile $I_{M^n}(\cdot)$ is difficult to compute. It is also difficult to estimate isoperimetric profile in terms of curvature and other geometric data. However, some known examples of symmetric spaces indicate that $I_{M^n}(\cdot)$ may depend on its sectional curvature K_{M^n}.

For example, on the n-dimensional Euclidean space $\mathbb{R}^n$, the classical isoperimetric inequality says that if $\Omega \subset \mathbb{R}^n$ is a compact domain with smooth boundary $\partial\Omega$, then

$$\mathrm{vol}_{n-1}(\partial\Omega) \geqslant c_n(\mathrm{vol}(\Omega))^{(n-1)/n}$$

where $\mathrm{vol}_{n-1}(\partial\Omega)$ denotes the $(n-1)$-dimensional volume of $\partial\Omega$, $\mathrm{vol}(\Omega)$ denotes the volume of Ω, and

$$c_n = \frac{\mathrm{vol}_{n-1}(S^{n-1}(1))}{\mathrm{vol}(B^n(1))^{(n-1)/n}}.$$

Hence, we have

$$I_{\mathbb{R}^n}(v) = c_n v^{(n-1)/n}. \tag{0.1}$$

If the sectional curvature satisfies $K_{M^n} \leqslant -1$ and M^n is simply connected, then

$$\mathrm{vol}_{n-1}(\partial\Omega) \geqslant (n-1)\,\mathrm{vol}(\Omega)$$

Benjamini was supported by the U.S. Army Research Office through the Mathematical Sciences Institute of Cornell University. Cao is supported in part by National Science Foundation grants DMS-9102212 and DMS 93-03711.

Received 7 October 1994. Revision received 2 October 1995.