Vol. 85, No. 1 DUKE MATHEMATICAL JOURNAL © October 1996

SOBOLEV INEQUALITIES AND MYERS'S DIAMETER THEOREM FOR AN ABSTRACT MARKOV GENERATOR

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1. Introduction. The classical theorem of (Bonnet-) Myers on the diameter [M] (see [C], [GHL]) states that if (M, g) is a complete, connected Riemannian manifold of dimension n ($\geqslant 2$) such that $\mathrm{Ric} \geqslant (n-1)g$, then its diameter $D = D(M)$ is less than or equal to π (and, in particular, M is compact). Equivalently, after a change of scale, if $\mathrm{Ric} \geqslant Rg$ with $R > 0$, and if S_r^n is the sphere of dimension n and constant curvature $R = (n-1)/r^2$ where $r > 0$ is the radius of S_r^n, then the diameter of M is less than or equal to the diameter of S_r^n, that is,

$$D \leqslant \pi r = \pi \sqrt{\frac{n-1}{R}}. \tag{1.1}$$

The aim of this work is to prove an analogue of Myers's theorem for an abstract Markov generator and to provide at the same time a new analytic proof of this result based on Sobolev inequalities. In particular, we will show how to get exact bounds on the diameter in terms of the Sobolev constant. As an introduction, let us describe the framework, referring to [B2] for further details. On some probability space $(E, \mathscr{E}, \mu)$, let L be a Markov generator associated to some semigroup $(P_t)_{t \geqslant 0}$ continuous in $\mathrm{L}^2(\mu)$. We will assume that L is invariant and symmetric with respect to μ, as well as ergodic. We assume furthermore that we are given a nice algebra $\mathscr{A}$ of bounded functions on E, containing the constants and stable by L and P_t (though this last hypothesis is not really needed but is used here for convenience) and by the action of C^∞ functions. We may then introduce, following P.-A. Meyer, the so-called *carré du champ* operator Γ as the symmetric bilinear operator on $\mathscr{A} \times \mathscr{A}$ defined by

$$2\Gamma(f,g) = \mathrm{L}(fg) - f\mathrm{L}g - g\mathrm{L}f, \qquad f, g \in \mathscr{A},$$

as well as the iterated carré du champ operator Γ_2

$$2\Gamma_2(f,g) = \mathrm{L}\Gamma(f,g) - \Gamma(f, \mathrm{L}g) - \Gamma(g, \mathrm{L}f), \qquad f, g \in \mathscr{A}.$$

Finally, we assume that L is a diffusion: for every C^∞ function Ψ on $\mathbb{R}^k$, and every finite family $F = (f_1, \ldots, f_k)$ in $\mathscr{A}$,

$$\mathrm{L}\Psi(F) = \nabla\Psi(F) \cdot \mathrm{L}F + \nabla\nabla\Psi(F) \cdot \Gamma(F, F).$$

Received 4 November 1994.