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STACKS OF STABLE MAPS AND GROMOV-WITTEN INVARIANTS

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0. Introduction. Let V be a projective algebraic manifold. In [15], Section 2, Gromov-Witten invariants of V were described axiomatically as a collection of linear maps

$$I^V_{g,n,\beta}\colon\ H^*(V)^{\otimes n} \to H^*(\overline{M}_{g,n}, \mathbb{Q}), \qquad \beta \in H_2(V, \mathbb{Z})$$

satisfying certain axioms, and a program to construct them by algebro-geometric (as opposed to symplectic) techniques was suggested. The program is based upon Kontsevich's notion of a stable map $(C, x_1, \dots, x_n, f)$, $f\colon C \to V$. This data consists of an algebraic curve C with n labeled points on it and a map f such that if an irreducible component of C is contracted by f to a point, then this component together with its special points is Deligne-Mumford stable. For more details, see [14] and below.

The construction consists of three major steps.

A. Construct an orbispace (or rather a stack) of stable maps $\overline{M}_{g,n}(V, \beta)$ such that $g =$ genus of C, $f_*([C]) = \beta$, and its two morphisms to V^n and $\overline{M}_{g,n}$. On the level of points, these morphisms are given, respectively, by

$$p\colon (C, x_1, \dots, x_n, f) \mapsto (f(x_1), \dots, f(x_n)),$$

$$q\colon (C, x_1, \dots, x_n, f) \mapsto [(C, x_1, \dots, x_n)]^{\text{stab}},$$

where the last expression means the stabilization of $(C, x_1, \dots, x_n)$.

B. Construct a "virtual fundamental class" $[\overline{M}_{g,n}(V, \beta)]_{\text{virt}}$, or "orientation" (see Definition 7.1 below), and use it to define a correspondence in the Chow ring $C^V_{g,n,\beta} \in A(V^n \times \overline{M}_{g,n})$.

This step suggested in [14] is quite subtle and will be dealt with elsewhere (see [3], [2]). It can be bypassed for $g = 0$ and $V = G/P$ (generalized flag spaces) where the virtual class coincides with the usual one (see [15]).

In general, it involves a definition of a new $\mathbb{Z}$-graded supercommutative structure sheaf on $\overline{M}_{g,n}(V, \beta)$. The virtual class is obtained as a product of the class of

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