Vol. 84, No. 3 DUKE MATHEMATICAL JOURNAL © September 1996

ON THE PRESCRIBED SCALAR CURVATURE PROBLEM ON 4-MANIFOLDS

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1. Introduction. Let (M^4, g_0) be a compact 4-dimensional Riemannian manifold with a nonnegative scalar curvature R_{g_0}. Let K be a C^3 positive function on M^4. The Kazdan-Warner problem [13] is the problem of finding suitable conditions on K such that K is the scalar curvature for some metric g on M^4 conformally equivalent to g_0. The metric g then reads

$$g = u^2 g_0,$$

where u is a positive function on M^4, satisfying the partial differential equation

$$(\mathbf{P}) \qquad \begin{cases} -L_{g_0} u = K(x) u^3 \\ u > 0 \end{cases}$$

where $L_{g_0} = \Delta - (1/6)R_{g_0}$ is the conformal Laplacian of M^4.

This problem has been studied in various works previous to ours, in dimension 2 and 3 (see [10], [8], [4], [12], [21], [11], [7]) as well as in high dimensions under more stringent conditions [2], [9], [15]. However, a new phenomenon appears in dimension $n \geqslant 5$, due to the fact that the self-interaction of the functions failing the Palais-Smale condition dominates the interaction of two of those functions [2]. While in dimension 2, 3 the reverse happens. In dimension 4, we have a *balance phenomenon*; that is, we show, in this paper, that the self-interaction and the interaction are of the same size.

This leads to an interesting new phenomenon, with a typical result. The expansion near infinity somewhat reminds us of the expansion of the Yamabe-type problem when several δ-functions (those failing the Palais-Smale condition) are involved. A natural Euler-Poincaré characteristic argument, very much different from the dimension 2 and 3 argument, allows us to derive the existence of a solution for this problem.

Another feature of the result is the fact that it extends to any compact Riemannian 4-dimensional manifold, without any restriction. It provides a topological condition under which K is the scalar-curvature of a metric conformal to the standard one. Examples of K's satisfying such a condition are very easy to

Received 14 January 1995. Revision received 7 December 1995.