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ON THE FOURIER COEFFICIENTS OF NONHOLOMORPHIC HILBERT MODULAR FORMS OF HALF-INTEGRAL WEIGHT

KAMAL KHURI-MAKDISI

Introduction. Let $f(z) = \sum_{n \geqslant 1} a_n e^{2\pi i n z}$ be a Hecke eigenform of half-integral weight $m + 1/2$, and let $g(z) = \sum_{n \geqslant 1} b_n e^{2\pi i n z}$ be the corresponding even-weight form, in the sense of [Sh1]. In particular, g has weight $2m$ and belongs to the same eigenvalues of Hecke operators as f. If $n = q^2 r$ with squarefree r, then a_n is expressible in terms of a_r and the $\{b_j\}$. At the end of [Sh2], Shimura suggested that a_r should be related to special values of Dirichlet series associated to g. This was borne out in [Wa], where Waldspurger proved the striking relation that for squarefree r, a_r^2 is essentially proportional to $\sum_{n \geqslant 1} \overline{\varphi_r}(n) b_n n^{-s}|_{s=m}$. Here we have twisted the standard Dirichlet series for g by a character φ_r obtained from the character of f and the quadratic character $(\frac{r}{\cdot})$.

The purpose of this paper is to derive generalizations of Waldspurger's relation, with an explicit proportionality constant, in the case where f and g are nonholomorphic Hilbert modular forms over a totally real number field F. If $F = \mathbf{Q}$, such forms are also called Maass forms. The method of proof, which ought to generalize to arbitrary number fields, follows, with some simplifications, that in [Sh10], which treats the case of holomorphic Hilbert modular forms. Previous investigations into this topic have induced work by Kohnen and Zagier ([KoZa] and [Ko]) in the holomorphic case, and Katok and Sarnak [KaSa] in the nonholomorphic case. Both of these treatments deal only with forms on the upper half-plane (i.e., $F = \mathbf{Q}$), with some additional restrictions. Recent (not yet published) work of M. Furusawa suggests that the method in [Sh10] and in this paper should generalize to yield a similar formula, in the case of the correspondence between automorphic forms on $Sp(n)$ and on $O(2n + 1)$.

Extending Shimura's work in [Sh10] to the nonholomorphic case involves two main difficulties. First, as the Fourier expansions of Maass forms involve Whittaker functions instead of exponentials, Mellin transforms and Rankin-Selberg convolutions produce more complicated "Gamma-factors" than usual; these factors must be explicitly evaluated, in order to yield precise versions of Waldspurger's relation. Second, whereas the Fourier expansions of holomorphic forms are indexed only by totally positive elements of the field F, the expansions of Maass forms are indexed by field elements of arbitrary signature; this makes the calculations rather more delicate. Section 3 of this paper explains in explicit detail how one overcomes both of these problems in constructing a Dirichlet

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