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COMPACT RIEMANN SURFACES WITH MANY SYSTOLES

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1. Introduction. The well-known lattice sphere problems in Euclidean spaces, namely, the best packing and the best kissing number (see [2]), have non-Euclidean analogues in dimension 2. They correspond to finding the Riemann surface of constant curvature -1 of a fixed signature with the systole of maximal length and with the maximal number of systoles, respectively. Surfaces with systoles of maximal length are treated in [7] and [8]. The most natural and most interesting examples are the surfaces corresponding to the principal congruence subgroups of $PSL(2, \mathbf{Z})$. On the other hand, almost no surface with the maximal number of systoles has been found until now; however, surfaces are known with many systoles. In [6] I proved the following theorem.

THEOREM [6]. *Let K be a positive integer. Then there exists a Riemann surface M_K corresponding to a principal congruence subgroup of $PSL(2, \mathbf{Z})$ so that M_K has more systoles than $K \cdot \dim(T(M_K))$ where $T(M_K)$ is the Teichmüller space of M_K.*

Similar surfaces can be constructed with other congruence subgroups of $PSL(2, \mathbf{Z})$; see [6]. These surfaces, however, all have cusps. The case of compact surfaces is treated in this paper. I shall prove the following theorem.

THEOREM. *Let K be a positive integer. Let $p \equiv 3 \bmod (4)$ be a prime. Then there exists a closed Riemann surface M_K of genus g corresponding to a congruence subgroup of the Fuchsian arithmetic group derived from the quaternion algebra $\frac{(p, -1)}{\mathbf{Q}}$ so that M_K has more systoles than $K(6g - 6)$.*

Notice that the size of the automorphism group of these surfaces is not responsible for the big number of systoles since this size is bounded by $14(6g - 6)$ (this is Hurwitz's theorem). Therefore, this result is rather surprising. For genus 2, 3, 4, and 5, for example, there are no closed surfaces known with more than $12g - 12$ systoles, and they probably do not exist; compare [8] and [9].

The proof of the theorem needs four steps. First, a way to classify the different systoles has to be found. It will be sufficient to consider only a part of the systoles, namely, those which contain a fixed point of an involution of the surface. I shall call them *canonical systoles.* The second step is the construction of as many different canonical systoles as one needs. The automorphism group of the surface

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