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COMPACTNESS OF ISOSPECTRAL CONFORMAL METRICS AND ISOSPECTRAL POTENTIALS ON A 4-MANIFOLD

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1. Introduction. In this article, we want to find some spectral invariant conditions which lead to the compactness of either isospectral conformal metrics or isospectral potentials. Let us recall that, for the first case, two Riemannian metrics g and g' on a compact manifold are said to be isospectral if their associated Laplace operators have identical spectra. For the second case, two potentials q, $\tilde{q} \in C^{\infty}(M)$ on a compact Riemannian manifold are said to be isospectral, i.e., $\tilde{q} \in \mathrm{Is}(q)$, if two Schrödinger operators $-\Delta + q$ and $-\Delta + \tilde{q}$ have the same spectra.

It is natural to ask to what extent the spectrum determines the metric in the first case, or the potential in the second case. Let us recall some work done on the extensively studied problem in the first case; we refer readers to them for more comprehensive references. For Riemann surfaces, Osgood-Phillips-Sarnak [OPS] and Wolpert [W] prove that the set of isospectral metrics is a compact set in the C^{∞}-topology. For manifolds of dimension n; $n \geqslant 3$, Brooks-Gordon [BG] and Brooks-Perry-Yang [BPY] show that there are many pairs of nonisometric isospectral conformal metrics. Brooks-Perry-Yang [BPY] and Chang-Yang [CY1], [CY2] prove the C^{∞}-compactness of the isospectral conformal metrics, and Anderson [An] proves the C^{∞}-compactness for the set of isospectral metrics with the length of their shortest closed geodesics being bounded uniformly for 3-dimensional manifolds. In dimension 4, Branson-Chang-Yang [BCY] prove the H^1-compactness for the conformal Laplace and Dirac operators, and the second author [X1], [X2] obtains C^{∞}-compactness for the isospectral metrics with some uniform boundedness on the L^4-norm of scalar curvature, or on the L^2-norm boundedness of the mean of the scalar curvature. Finally, for H^1-compactness of the isospectral metrics, Gursky [Gu] proves, for all dimensions $n \geqslant 4$, the set is H^1-compact if there is a uniform boundedness on the L^p, $p > n/2$, norm of the full curvature tensor. For the second case, Guillemin [Gm] shows that certain potentials on certain manifolds are spectrally rigid, i.e., for any two isospectral potentials q and $\tilde{q}$, there is an isometry ψ of M such that $\tilde{q} = q \circ \psi$, whereas Lax [L] shows that the periodic solutions of Korteweg-de Vries equation provide nontrivial isospectral deformations for potentials on S^1. For the compactness, Brüning [B] proves that the set of

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