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NONINVOLUTORY HOPF ALGEBRAS AND 3-MANIFOLD INVARIANTS

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This paper presents a definition of an invariant $\#(M, H)$ which depends on a framed, closed 3-manifold M and a finite-dimensional Hopf algebra H, and whose value lies in the ground field of H. The Hopf algebra H need not be quasitriangular, triangular, ribbon, modular, a quantum deformation, involutory, or semisimple, nor does it need to have any other decoration or structural property. It can be any finite-dimensional example of the object defined by Sweedler [16] and Drinfel'd [5] or, more generally, a finite-dimensional Hopf superalgebra or a Hopf object in any category which sufficiently resembles the category of finite-dimensional vector spaces. In a previous paper [8], the author defined $\#(M, H)$ for involutory Hopf algebras (Hopf algebras in which the square of the antipode is the identity) and for closed and oriented but unframed 3-manifolds. An important intermediate class of finite-dimensional Hopf algebras is the class of balanced Hopf algebras, for which the 3-manifold M need only be oriented and combed rather than framed. Recall that a framing of a 3-manifold is a homotopy class of linearly independent triples of tangent vectors fields. A combing is defined as the homotopy class of a single nonvanishing tangent vector field.

In a subsequent paper [7], we will define the related invariant $\#(M, L, H)$, where M is a framed, closed 3-manifold, H is a Hopf algebra, and L is a framed link in M. More generally, the invariant $\#(M, G, H)$ can be defined, where G is a framed graph in M. When $M = \mathbb{S}^3$, these invariants coincide with the Reshetikhin-Turaev invariants of links and ribbons graphs derived from $D(H)$, the quantum double of H. In particular, if q is a root of unity and $\mathfrak{g}$ is a simple Lie algebra, the Reshetikhin-Turaev invariants for the finite-dimensional quantum groups $u_q(\mathfrak{g})$ yield root-of-unity values of the familiar quantum link invariants, such as the Jones polynomial, the HOMFLY polynomial, the Kauffman polynomial, and the quantum G_2 link invariant. The Hopf algebra $u_q(\mathfrak{g})$ is almost the quantum double of $u_q(\mathfrak{g}^+)$, a truncated quantum deformation of (the enveloping algebra) $U(\mathfrak{g}^+)$, where $\mathfrak{g}^+$ is a Borel subalgebra of $\mathfrak{g}$. Therefore $H = u_q(\mathfrak{g}^+)$ is an important special case of the invariant $\#(M, H)$ that we define here.

Some other important special cases of $\#(M, H)$ are the following: $\#(\mathbb{S}^3, H) = 1$ by normalization, while $\#(\mathbb{S}^2 \times \mathbb{S}^1, H) = \mathrm{Tr}(S^2)$ is $\dim H$ when H is involutory and 0 when H is noninvolutory, and $\#(\mathbb{R}P^3, H) = \mathrm{Tr}(S)$. (Here S is the

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