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ON SETS OF CRITICAL VALUES IN THE REAL LINE

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A compact subset $B \subset \mathbb{R}$ equals the critical value set of some differentiable function $f: \mathbb{R} \to \mathbb{R}$ if and only if B has Lebesgue measure zero (see [H], [N1]). In many cases, however, the function f cannot be very smooth. If, for example, the standard middle-thirds Cantor set $\mathscr{C}$ were the critical value set of a C^2 function $f: \mathbb{R} \to \mathbb{R}$, then the set sum $\mathscr{C} + \mathscr{C} = [0, 2]$ would lie within the critical value set of the function $F: \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x, y) = f(x) + f(y)$. By the classical Morse-Sard theorem, however, the critical value set of *any* C^2 function $\mathbb{R}^2 \to \mathbb{R}$ must have measure zero.

In this paper, we address the problem of what further conditions are necessary to characterize those compact sets that are equal to the set of critical values $\mathrm{cv}(f)$ for some function f belonging to a particular smoothness class. We limit ourselves in this paper to the one-dimensional case of functions $f: \mathbb{R} \to \mathbb{R}$. Since these are essentially local matters, it is convenient to focus on functions with compactly supported derivatives. We make the following definition.

Definition. Given a class $\mathscr{C}$ of functions in $C^1(\mathbb{R}, \mathbb{R})$, denote by $CV(\mathscr{C})$ the collection of all sets B of the form $B = \mathrm{cv}(f)$ for some $f \in \mathscr{C}$ with compactly supported derivative.

The statement of our main results requires the notion of *gap sum.* The bounded components (if any) of the complement of a compact set $B \subset \mathbb{R}$ will be called the *gaps* of B. Any such gap is an open interval with endpoints in B. The collection of all gaps of B is denoted $\mathscr{G}_B$, and for real $\alpha > 0$, the degree-α gap sum of B is then defined as

$$G_\alpha(B) = \sum_{I \in \mathscr{G}_B} |I|^\alpha,$$

where $|I|$ denotes the diameter of the gap I.

For $k \in \mathbb{Z}_+$ and $\lambda \in [0, 1)$, we denote by $C^{k+\lambda}$ the space of C^k functions $\mathbb{R} \to \mathbb{R}$ whose kth derivative satisfies a local λ-Hölder condition. We say a set is *1-null* if it has Lebesgue measure zero. One object of this paper is to prove the following theorem.

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