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EXTREMAL LENGTH ESTIMATES AND PRODUCT REGIONS IN TEICHMÜLLER SPACE

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1. Introduction. There is a longstanding but imperfect analogy between the geometry of the Teichmüller space $\mathcal{T}(S)$ of a surface and that of a complete, negatively curved space. For example, $\mathcal{T}(S)$ admits a boundary at infinity similar to that of hyperbolic space (see, e.g., [8], [23]), and the Teichmüller geodesic flow on its quotient, the moduli space, is ergodic (see [22]). This paper studies one of the strong ways in which this analogy fails, namely, the existence of large regions in the space which are closely approximated by products of lower-dimensional metric spaces.

There are several natural metrics on $\mathcal{T}(S)$; we will consider throughout only the *Teichmüller metric* (see Section 2.5). The first indication of positively curved behavior of the Teichmüller metric came from Masur [20], in which examples were given of geodesic rays with a common basepoint, which remain a bounded distance apart for all time. This contradicts nonpositive curvature on small scales at every point (although it also involves large-scale behavior of the geodesics in a subtle way). More recently, Masur and Wolf [24] have given examples of geodesic triangles which fail, arbitrarily badly, the "thin triangle" condition for hyperbolicity in the sense of Gromov and Cannon [5], [6], [12]; this implies that the large-scale behavior of the metric is not negatively curved. The main result of this paper describes regions of $\mathcal{T}(S)$ where the larger-scale behavior exhibits some characteristics of positive curvature.

The main theorem can be summarized as follows. (A complete statement appears in Section 6.) Let $\mathcal{T}(S)$ be the Teichmüller space of a surface S of finite type, endowed with the Teichmüller metric. Let $\gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a system of disjoint, homotopically distinct simple closed curves on S, and let $\mathrm{Thin}_\varepsilon(S, \gamma)$ denote the set of $\sigma \in \mathcal{T}(S)$ for which $\ell_\sigma([\gamma_i]) \leqslant \varepsilon$ for all i, where $\ell_\sigma([\gamma_i])$ denotes hyperbolic length of the σ-geodesic representative of γ_i. Let X_γ denote the product space $\mathcal{T}(S\backslash\gamma) \times \mathbf{H}_1 \times \cdots \times \mathbf{H}_k$, where $S\backslash\gamma$ is considered as a punctured surface, and each $\mathbf{H}_i$ is a copy of the hyperbolic plane. Endow X_γ with the *sup metric*, $d_X = \max\{d_{\mathcal{T}(S\backslash\gamma)}, d_{\mathbf{H}_1}, \ldots, d_{\mathbf{H}_k}\}$, of the metrics on the factors.

THEOREM 6.1 (Summary). *The Fenchel-Nielsen coordinates on $\mathcal{T}(S)$ give rise to a natural homeomorphism $\Pi\colon \mathcal{T}(S) \to X_\gamma$. For ε sufficiently small, this homeomorphism restricted to* $\mathrm{Thin}_\varepsilon(S, \gamma)$ *distorts distances by a bounded additive amount.*

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