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THE BEURLING-SELBERG EXTREMAL FUNCTIONS FOR A BALL IN EUCLIDEAN SPACE

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1. Introduction. In this paper we investigate a type of extremal problem which originates in work of A. Beurling. One of our objectives is to show how Beurling's extremal problem can be reformulated and solved within the theory of Hilbert spaces of entire functions as developed by L. de Branges. To begin with, however, we describe an important special case of the extremal problem and some of its applications. Let ξ, δ, and ν be real numbers with $0 < \delta$ and $-1 < \nu$. We consider the problem of determining real entire functions $S(z)$ and $T(z)$ such that

(1.1) $S(x) \leqslant \operatorname{sgn}(x - \xi) \leqslant T(x)$ for all real x,

(1.2) $S(z)$ and $T(z)$ are entire functions of exponential type at most $2\pi\delta$,

and the value of the integral

$$\frac{1}{2}\int_{-\infty}^{\infty} \{T(x) - S(x)\}|x|^{2\nu+1}\, dx \tag{1.3}$$

is as small as possible. Let $u_\nu(\xi, \delta)$ denote the infimum of the integral (1.3) taken over the set of all pairs of real entire functions $S(z)$ and $T(z)$ which satisfy (1.1) and (1.2). We will give an explicit formula for $u_\nu(\xi, \delta)$ as a finite combination of Bessel functions and determine the unique pair of extremal functions which minimize the integral (1.3).

THEOREM 1. *Let ξ, δ and ν be real numbers with $0 < \delta$ and $-1 < \nu$. Then $u_\nu(\xi, \delta)$ satisfies:*

(i) $u_\nu(\xi, \delta) = u_\nu(-\xi, \delta)$,
(ii) *if* $0 < \kappa$ *then* $u_\nu(\xi, \delta) = \kappa^{2\nu+2}u_\nu(\kappa^{-1}\xi, \kappa\delta)$,
(iii) $u_\nu(0, \delta) = \Gamma(\nu + 1)\Gamma(\nu + 2)(2/\pi\delta)^{2\nu+2}$,
(iv) *if* $0 < \xi$ *then*

$$u_\nu(\xi, \pi^{-1}) = 2\xi^{2\nu+1}\{\xi J_\nu(\xi)^2 + \xi J_{\nu+1}(\xi)^2 - (2\nu + 1)J_\nu(\xi)J_{\nu+1}(\xi)\}^{-1}, \tag{1.4}$$

where $J_\nu(z)$ is the Bessel function.

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