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# THE GEOMETRY OF DEGREE-4 CHARACTERISTIC CLASSES AND OF LINE BUNDLES ON LOOP SPACES II

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**1. Introduction.** In this paper, we continue the study of degree-4 characteristic classes begun in Part I [9]. The underlying theme is to find sheaf-theoretic objects which represent classes $\alpha \in H^4(BG; \mathbb{Z})$ and to explore their geometry. Such an object is a "sheaf of bicategories" and is an example of what L. Breen has called a 2-gerbe [5], [6]. The situation we are dealing with is entirely analogous to that which exists between the first Chern class of a line bundle and the differential geometry of the line bundle itself. We will assume throughout that the reader is familiar with Part I, where this program was carried out in the case of a compact 1-connected Lie group $G$ and its complexification $G_{\mathbb{C}}$.

We begin in Section 2 with the case of the circle $S^1$ and its complexification $\mathbb{C}^*$. Here we find an explicit 2-gerbe (together with a "notion of connectivity") representing the square of the universal first Chern class $c_1^2$ (Theorem 2.4 and Remark 2.5). This is done by generalizing Deligne's observation that the construction of a holomorphic line bundle-with-connection from two invertible holomorphic functions can be interpreted geometrically as a cup product [2], [14].

In Section 3, we consider the the natural transgression $\tau: H^4(B\mathbb{C}^*) \to H^3(LB\mathbb{C}^*)$ to the free loop space. This corresponds geometrically to taking the holonomy of the 2-gerbe associated to $c_1^2$ around a loop. From the Segal-Witten reciprocity law (Theorem 5.9 of Part I) specialized to the case of $\mathbb{C}^*$, we know that $\tau$ singles out those extensions of $L\mathbb{C}^*$ by $\mathbb{C}^*$ which have the reciprocity property; these are the extensions that split canonically over loops which extend holomorphically to the interior of any Riemann surface. The main point of Section 3 is to prove that this reciprocity law implies the classical reciprocity theorem of Weil; let $f, g$ be any two meromorphic functions on a Riemann surface with disjoint zeroes and poles, then [14], [26]

$$\prod_p f(p)^{\operatorname{ord} g(p)} = \prod_p g(p)^{\operatorname{ord} f(p)}.$$

While this was certainly known to Witten [42] and Segal [36], we feel that our approach using gerbes is the right framework to understand this phenomenon. Indeed the reciprocity law is exactly what is needed to fill in a 2-arrow between two given 1-arrows in a 2-gerbe.

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